ENGINEERING FORMULAS

Compiled By

Engineer R Hasan

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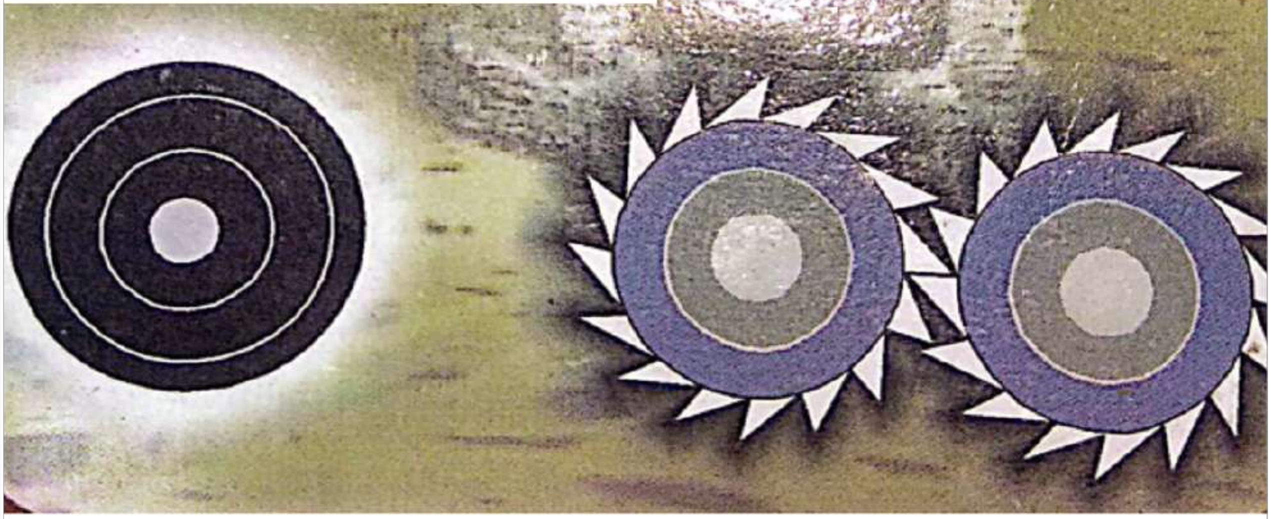
Research Scholar, Languages Literatures

Born in Pakistan, HSK-3 Chinese, Goethe Institut Teilnahmebestatigung B1 datum 12.2020 - Research Scholar Alumni of EIEPC | Engineer's Industrial Education & Practical Center Master of Engineering NED University of Technology, Karachi, Pakistan.



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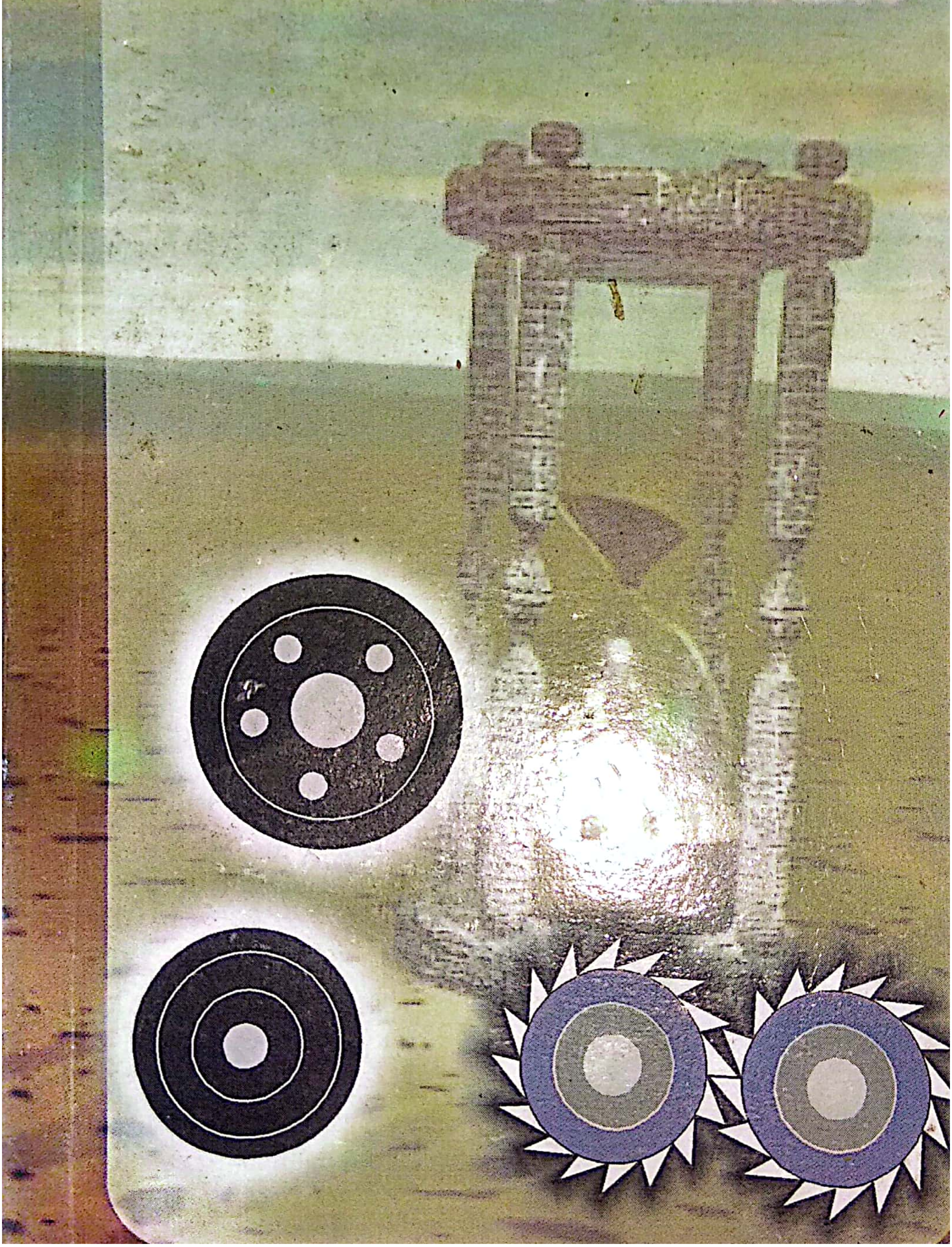

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Preface

The purpose of this collection of technical formulae is to provide a brief, clear and handy guide to the more important technical and mathematical formulae.

Since the book has been printed on one side of the page only, the facing pages are available for additional notes.

Each separate subject has been associated with a capital letter. The various formulae have been grouped under corresponding small letters and numbered serially. This method enables the formulae used in any particular calculation to be indicated.

Preface

to the 2nd edition

"Engineering Formulas" has been revised to include S.I. Units and in addition, several sections have been extended and brought up to date.

Method of Presentation and Use of Units

Most of the equations clearly reveal the physical relationships which they describe and are valid regardless of the system of units employed, provided that they are consistent.

Some of the equations are empirical in origin and the units quoted must be used in the formula to obtain the correct result, these are mainly to be found in sections Q and R.

It is intended that the Stroud notation is used when evaluating the formulae i.e. both the quantity and the unit is substituted for a given symbol and the subsequent calculation involves manipulation of numbers and units together.

For example, taking equation I 23: $t = \frac{s}{v}$

if s (distance) $= 2{\cdot}8$ metres
v (speed) $= 8$ metres/second

then $t = \dfrac{2{\cdot}8 \text{ metres} \times \text{second}}{8 \text{ metres}}$

hence $t = 0{\cdot}35$ seconds (time)
cancelling the unit 'metres'

It is clear that t should have the units of time; if it does not, then it is obvious that an error has been made and the working should be checked. As a help, in many cases, the anticipated units are quoted using the abbreviation "EU", Example-Unit.

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When the numerical values and the units are included in the calculations, their equivalents or definitions are best written so that they are dimensionless and have the value of 1·0. In this form they are sometimes called "Unity Brackets" and their use can be illustrated in three ways:

with consistent units,

equation a 1,

$$1 \text{ km} = 10^3 \text{ m} \qquad \text{becomes} \qquad 1 = \left[\frac{\text{km}}{10^3 \text{ m}}\right]$$

equation a 58

$$12 \text{ in} = 1 \text{ ft} \qquad \text{becomes} \qquad 1 = \left[\frac{1 \text{ ft}}{12 \text{ in}}\right]$$

equation a 86

$$778{\cdot}6 \text{ ft lbf} = 1 \text{ Btu} \qquad \text{becomes} \qquad 1 = \left[\frac{778{\cdot}6 \text{ ft lbf}}{\text{Btu}}\right]$$

for example, to convert 14·7 lbf/in² to lbf/ft²

$$14{\cdot}7 \, \frac{\text{lbf}}{\text{in}^2} = 14{\cdot}7 \, \frac{\text{lbf}}{\text{in}^2} \left[\frac{12 \text{ in}}{\text{ft}}\right]^2 = 14{\cdot}7 \times 144 \, \frac{\text{lbf}}{\text{ft}^2} = 2117 \, \frac{\text{lbf}}{\text{ft}^2}$$

in the conversion between different systems of units,

equation a 36

$$1 \text{ kgf} = 9{\cdot}81 \text{ N} \qquad \text{becomes} \qquad 1 = \left[\frac{9{\cdot}81 \text{ N}}{\text{kgf}}\right]$$

equation a 61

$$1 \text{ m} = 3{\cdot}281 \text{ ft} \qquad \text{becomes} \qquad 1 = \left[\frac{\text{m}}{3{\cdot}281 \text{ ft}}\right]$$

equation a 108

$$1 \text{ Btu/lb} = 0{\cdot}556 \text{ kcal/kg} \quad \text{becomes} \quad 1 = \left[\frac{0{\cdot}556 \text{ kcal lb}}{\text{kg Btu}}\right]$$

For example, to convert 1000 kgf/cm² to S.I. units,

$$1000 \, \frac{\text{kgf}}{\text{cm}^2} = 1000 \, \frac{\text{kgf}}{\text{cm}^2} \left[\frac{9{\cdot}81 \text{ N}}{\text{kgf}}\right] \left[\frac{10^4 \text{ cm}^2}{\text{m}^2}\right] \left[\frac{\text{MN}}{10^6 \text{ N}}\right]$$

$$= 98{\cdot}1 \, \frac{\text{MN}}{\text{m}^2}$$

in the use of definitions:

1. lbf is the force required to accelerate a mass of 1 lb at the rate of 32·174 ft/s².

$$1\ \text{lbf} = 1\ \text{lb} \times 32{\cdot}174\ \frac{\text{ft}}{\text{s}^2} \qquad \text{becomes} \qquad 1 = \left[\frac{32{\cdot}174\ \text{lb ft}}{\text{s}^2\ \text{lbf}}\right].$$

Similarly, the Newton is defined by the equation

$$1\ \text{N} = 1\ \text{kg} \times \frac{1\ \text{m}}{\text{s}^2} \qquad \text{which becomes} \qquad 1 = \left[\frac{\text{N s}^2}{\text{kg m}}\right]$$

and

$$1\ \text{kgf} = 1\ \text{kg} \times 9{\cdot}81\ \frac{\text{m}}{\text{s}^2} \qquad \text{becomes} \qquad 1 = \left[\frac{9{\cdot}81\ \text{kg m}}{\text{kgf s}^2}\right]$$

For example, to find the force in S.I. units required to accelerate a mass of 3 lb at the rate of 2·5 ft/s², proceed as follows:

$$F = m\,a, \ \text{equation m 1.}$$

$$\therefore F = 3\ \text{lb} \times 2{\cdot}5\ \frac{\text{ft}}{\text{s}^2}\left[\frac{0{\cdot}4536\ \text{kg}}{\text{lb}}\right]\left[\frac{\text{m}}{3{\cdot}281\ \text{ft}}\right]\left[\frac{\text{N s}^2}{\text{kg m}}\right]$$

$$= \frac{3 \times 2{\cdot}5 \times 0{\cdot}4536}{3{\cdot}281}\ \text{N} = 1{\cdot}036\ \text{N}$$

which is a unit of force.

Base Quantities and Base Units of the International System of Measurement

base quantity		base unit	
name	symbol (*italic* *letters*)	name	symbol (vertical letters)
length	*l*	metre	m
mass	*m*	kilogram	kg
time	*t*	second	s
electric current	*I*	ampere	A
absolute temperature	*T*	kelvin	K
light intensity	I_v	candela	cd

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LIST OF SYMBOLS

Space and time

a, b	distances
α, β, γ	angles
Ω	solid angle
l	length
b	width
h	height
r, R	radius
d, D	diameter (diagonal)
s	distance covered, perimeter
s	wall thickness
u, U	circumference
A	area, cross-section
A_m	generated surface
A_o	surface area
V	volume
t	time
v	velocity, linear
ω	velocity, angular
a	acceleration, linear
α	acceleration, angular
g	acceleration, gravitational
t	pitch

Periodical and related phenomena

T	period
f	frequency
n	rotational speed
ω	angular frequency
λ	wavelength
φ	phase angle

Mechanics

m	mass
ρ	density
v	specific volume
p	impulse
I	moment of inertia
F	force
G	gravitational force
M	moment
M_R	frictional torque
p	pressure (force on surface)
σ	tensile or compressive stress (normal stress)
τ	shear stress
ε	extension
γ	shear deflection (angular)
E	modulus of elasticity (Young's modulus)
G	modulus of rigidity
I	second moment of area
W	moment of resistance
H	static moment
μ	coefficient of sliding friction
μ_0	coefficient of static friction
μ_q	coefficient of frict. of a radial bearing
μ_l	coeff. of friction of a longitudinal bearing
η, μ, z	dynamic viscosity
ν	kinematic viscosity
W	work
P	power
η	efficiency

Heat

T	absolute temperature
t	temperature
α	linear coefficient of expansion
γ	cubic coefficient of expansion
ϕ	heat current or flow
φ	density of heat flow
q	quantity of heat per unit mass
Q	quantity of heat
c_p	specific heat at constant pressure
c_v	specific heat at constant volume
γ	ratio of c_p to c_v
R	gas constant
λ	thermal conductivity
α	heat transfer coefficient
k	coefficient of heat transmission
C	radiation constant
v	specific volume

Electricity and magnetism

I	current
J	current density
V, U	voltage
U_q	source voltage
R	resistance
G	conductance
Q	quantity of electricity (charge)
C	capacitance
D	dielectric displacement
E	electric field strength
ϕ	magnetic flux
B	magnetic induction
L	inductance
H	magn. field strength
Θ	circulation (magnetic potential)
V	magnetic voltage
R_m	magnetic resistance
Λ	magnetic conductance
δ	length of air gap
α	temperature coefficient of resistance
γ	conductivity
ρ	resistivity
ε	permittivity, dielectric constant
ε_0	absolute permittivity
ε_r	relative permittivity
N	number of turns
μ	permeability
μ_0	absolute permeability
μ_r	relative permeability
p	number of pairs of poles
z	number of conductors
Q	quality, figure of merit
δ	loss angle
Z	impedance
X	reactance
P_s	apparent power
P_q	reactive power
C_M	moment constant

Optics

c	velocity of light
I_v	luminous intensity
ϕ_v	luminous flux
Q	quantity of light
E_v	illumination
L_v	luminance, brightness
n	refractive index
D	refractive power
f	focal length
m	magnification, enlargement
s	distinct visual distance

Density of Different Liquids

MASS PER UNIT OF Volume

FRESH WATER = 1 TON
Sea water = 1.025
Diesel = 0.870
Lube = 0.865
Heavy fuel = 0.980

Units	A
Areas	B
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UNITS

A 1

Decimal multiples and fractions of units

da = deca = 10^{1}
h = hecto = 10^{2}
k = kilo = 10^{3}
M = mega = 10^{6}
G = giga = 10^{9}
T = tera = 10^{12}

d = deci = 10^{-1}
c = centi = 10^{-2}
m = milli = 10^{-3}
µ = micro = 10^{-6}
n = nano = 10^{-9}
p = pico = 10^{-12}
f = femto = 10^{-15}
a = atto = 10^{-18}

Units of length

		m	µm	mm	cm	dm	km
a 1	1 m =	1	10^{6}	10^{3}	10^{2}	10	10^{-3}
a 2	1 µm =	10^{-6}	1	10^{-3}	10^{-4}	10^{-5}	10^{-9}
a 3	1 mm =	10^{-3}	10^{3}	1	10^{-1}	10^{-2}	10^{-6}
a 4	1 cm =	10^{-2}	10^{4}	10	1	10^{-1}	10^{-5}
a 5	1 dm =	10^{-1}	10^{5}	10^{2}	10	1	10^{-4}
a 6	1 km =	10^{3}	10^{9}	10^{6}	10^{5}	10^{4}	1

Units of length (continued)

		mm	pm	nm	µm	Å	mÅ
a 7	1 mm =	1	10^{9}	10^{6}	10^{3}	10^{7}	10^{10}
a 8	1 pm =	10^{-9}	1	10^{-3}	10^{-6}	10^{-2}	10
a 9	1 nm =	10^{-6}	10^{3}	1	10^{-3}	10	10^{4}
a 10	1 µm =	10^{-3}	10^{6}	10^{3}	1	10^{4}	10^{7}
a 11	1 Å =	10^{-7}	10^{2}	10^{-1}	10^{-4}	1	10^{3}
a 12	1 mÅ =	10^{-10}	10^{-1}	10^{-4}	10^{-7}	10^{-3}	1

Units of area

		m^2	$µm^2$	mm^2	cm^2	dm^2	km^2
a 13	1 m^2 =	1	10^{12}	10^{6}	10^{4}	10^{2}	10^{-6}
a 14	1 $µm^2$ =	10^{-12}	1	10^{-6}	10^{-8}	10^{-10}	10^{-18}
a 15	1 mm^2 =	10^{-6}	10^{6}	1	10^{-2}	10^{-4}	10^{-12}
a 16	1 cm^2 =	10^{-4}	10^{8}	10^{2}	1	10^{-2}	10^{-10}
a 17	1 dm^2 =	10^{-2}	10^{10}	10^{4}	10^{2}	1	10^{-8}
a 18	1 km^2 =	10^{6}	10^{18}	10^{12}	10^{10}	10^{8}	1

Å = Ångström | 1 mÅ = XE (Röntgen-unit)

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UNITS

A 2

Units of volume

		m^3	mm^3	cm^3	$dm^3 = l$[2]	km^3
a 19	1 m^3 =	1	10^9	10^6	10^3	10^9
a 20	1 mm^3 =	10^{-9}	1	10^{-3}	10^{-6}	10^{-18}
a 21	1 cm^3 =	10^{-6}	10^3	1	10^{-3}	10^{-15}
a 22	1 dm^3 = 1 l =	10^{-3}	10^6	10^3	1	10^{-12}
a 23	1 km^3 =	10^9	10^{18}	10^{15}	10^{12}	1

Units of mass

		kg	mg	g	dt	t = Mg
a 24	1 kg =	1	10^6	10^3	10^{-2}	10^{-3}
a 25	1 mg =	10^{-6}	1	10^{-3}	10^{-8}	10^{-9}
a 26	1 g =	10^{-3}	10^3	1	10^{-5}	10^{-6}
a 27	1 dt =	10^2	10^8	10^5	1	10^{-1}
a 28	1 t = 1 Mg =	10^3	10^9	10^6	10	1

Units of time

		s	ns	µs	ms	min
a 29	1 s =	1	10^9	10^6	10^3	$16{\cdot}66 \times 10^{-3}$
a 30	1 ns =	10^{-9}	1	10^{-3}	10^{-6}	$16{\cdot}66 \times 10^{-12}$
a 31	1 µs =	10^{-6}	10^3	1	10^{-3}	$16{\cdot}66 \times 10^{-9}$
a 32	1 ms =	10^{-3}	10^6	10^3	1	$16{\cdot}66 \times 10^{-6}$
a 33	1 min =	60	60×10^9	60×10^6	60×10^3	1

Units of force (gravitational force also)

		N[1]	kN	MN	kgf	dyn
a 34	1 N =	1	10^{-3}	10^{-6}	$0{\cdot}102$	10^5
a 35	1 kN =	10^3	1	10^{-3}	$0{\cdot}102 \times 10^3$	10^8
a 36	1 MN =	10^6	10^3	1	$0{\cdot}102 \times 10^6$	10^{11}
a 37	1 kgf =	$9{\cdot}81$	$9{\cdot}81 \times 10^{-3}$	$9{\cdot}81 \times 10^{-6}$	1	$9{\cdot}81 \times 10^5$
a 38	1 dyn =	10^{-5}	10^{-8}	10^{-11}	$0{\cdot}102 \times 10^{-5}$	1

[1] 1 N = 1 kg m/s^2

[2] 1 l = 1 liter

UNITS

Units of pressure

		Pa = N/m²	N/mm²	bar	kgf/cm² = at	Torr [2]
a 39	1 Pa [1] =	1	10^{-6}	10^{-5}	$1{\cdot}02 \times 10^{-5}$	0·0075
a 40	1 N/mm² =	10^6	1	10	10·2	$7{\cdot}5 \times 10^3$
a 41	1 bar =	10^5	0·1	1	1·02	750
a 42	1 kgf/cm² =	98100	$9{\cdot}81 \times 10^{-2}$	0·981	1	736
a 43	1 Torr [2] =	133	$0{\cdot}133 \times 10^{-3}$	$1{\cdot}33 \times 10^{-3}$	$1{\cdot}36 \times 10^{-3}$	1

Units of work

		J = N m = W s	kW h	kgf m	kcal	PS h
a 44	1 J [3] =	1	$0{\cdot}278 \times 10^{-6}$	0·102	$0{\cdot}239 \times 10^{-3}$	$0{\cdot}378 \times 10^{-6}$
a 45	1 kW h =	$3{\cdot}60 \times 10^6$	1	367×10^3	860	1·36
a 46	1 kgf m =	9·81	$2{\cdot}72 \times 10^{-6}$	1	$2{\cdot}345 \times 10^{-3}$	$3{\cdot}70 \times 10^{-6}$
a 47	1 kcal =	4186	$1{\cdot}16 \times 10^{-3}$	426·9	1	$1{\cdot}58 \times 10^{-3}$
a 48	1 PS h =	$2{\cdot}65 \times 10^6$	0·736	$0{\cdot}27 \times 10^6$	632	1

Units of power

		W = J/s = N m/s	kW	kgf m/s	kcal/h	PS
a 49	1 W [4] =	1	10^{-3}	0·102	0·860	$1{\cdot}36 \times 10^{-3}$
a 50	1 kW =	1000	1	102	860	1·36
a 51	1 kgf m/s =	9·81	$9{\cdot}81 \times 10^{-3}$	1	8·43	$13{\cdot}3 \times 10^{-3}$
a 52	1 kcal/h =	1·16	$1{\cdot}16 \times 10^{-3}$	0·119	1	$1{\cdot}58 \times 10^{-3}$
a 53	1 PS =	736	0·736	75	632	1

Unit for mass

(used for jewels)

a 54 1 carat ≙ 200 mg = $0{\cdot}2 \times 10^{-3}$ kg = $\frac{1}{5000}$ kg

Unit of fineness

(used for precious metals)

a 55 24 carat ≙ 1000·00 ‰ | 18 carat ≙ 750·00 ‰

a 56 14 carat ≙ 583·33 ‰ | 8 carat ≙ 333·33 ‰

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[1] 1 Pa = 1 N/m²

[3] 1 J = 1 N m = 1 W s

[2] 1 Torr ≙ 1 mm Hg

[4] 1 W = 1 J/s = 1 N m/s

Conversion, Anglo-American to metric units

Units of length

		in	ft	yd	mm	m	km
a 57	1 in =	1	0·08333	0·02778	25·4	0·0254	–
a 58	1 ft =	12	1	0·3333	304·8	0·3048	–
a 59	1 yd =	36	3	1	914·4	0·9144	–
a 60	1 mm =	0·03937	3281×10^{-6}	1094×10^{-6}	1	0·001	10^{-6}
a 61	1 m =	39·37	3·281	1·094	1000	1	0·001
a 62	1 km =	39370	3281	1094	10^{6}	1000	1

Units of area

		sq in	sq ft	sq yd	cm²	dm²	m²
a 63	1 sq in =	1	–	–	6·452	0·06452	–
a 64	1 sq ft =	144	1	0·1111	929	9·29	0·0929
a 65	1 sq yd =	1296	9	1	8361	83·61	0·8361
a 66	1 cm^2 =	0·155	–	–	1	0·01	0·0001
a 67	1 dm^2 =	15·5	0·1076	0·01196	100	1	0·01
a 68	1 m^2 =	1550	10·76	1·196	10000	100	1

Units of volume

		cu in	cu ft	cu yd	cm³	dm³	m³
a 69	1 cu in =	1	–	–	16·39	0·01639	–
a 70	1 cu ft =	1728	1	0·037	28320	28·32	0·0283
a 71	1 cu yd =	46656	27	1	765400	–	–
a 72	1 cm^3 =	0·06102	3531×10^{-8}	$1·31 \times 10^{-6}$	1	0·001	10^{-6}
a 73	1 dm^3 =	61·02	0·03531	0·00131	1000	1	0·001
a 74	1 m^3 =	61023	3531	130·7	10^{6}	1000	1

Units of mass

		dram	oz	lb	g	kg	Mg
a 75	1 dram =	1	0·0625	0·003906	1·772	0·00177	–
a 76	1 oz =	16	1	0·0625	28·35	0·02835	–
a 77	1 lb =	256	16	1	453·6	0·4536	–
a 78	1 g =	0·5644	0·03527	0·002205	1	0·001	10^{-6}
a 79	1 kg =	564·4	35·27	2·205	1000	1	0·001
a 80	1 Mg =	$564·4 \times 10^{3}$	35270	2205	10^{6}	1000	1

continued A 5

UNITS

A 5

continued from A 4

Units of work

		ft lb	kgf m	J = W s	kW h	kcal	Btu
a 81	1 ft lb =	1	0·1383	1·356	$376{\cdot}8\times10^{-9}$	324×10^{-6}	$1{\cdot}286\times10^{-3}$
a 82	1 kgf m =	7·233	1	9·807	$2{\cdot}725\times10^{-6}$	$2{\cdot}344\times10^{-3}$	$9{\cdot}301\times10^{-3}$
a 83	1 J = 1 W s =	0·7376	0·102	1	$277{\cdot}8\times10^{-9}$	239×10^{-6}	$948{\cdot}4\times10^{-6}$
a 84	1 kW h =	$2{\cdot}655\times10^{6}$	$367{\cdot}1\times10^{3}$	$3{\cdot}6\times10^{6}$	1	860	3413
a 85	1 kcal =	$3{\cdot}087\times10^{3}$	426·9	4187	$1{\cdot}163\times10^{-3}$	1	3·968
a 86	1 Btu =	778·6	107·6	1055	293×10^{-6}	0·252	1

Units of power

		hp	kgf m/s	J/s = W	kW	kcal/s	BTU/s
a 87	1 hp =	1	76·04	745·7	0·7457	0·1782	0·7073
a 88	1 kgf m/s =	$13{\cdot}15\times10^{-3}$	1	9·807	$9{\cdot}807\times10^{-3}$	$2{\cdot}344\times10^{-3}$	$9{\cdot}296\times10^{-3}$
a 89	1 J/s = 1 W =	$1{\cdot}341\times10^{-3}$	0·102	1	10^{-3}	239×10^{-6}	$948{\cdot}4\times10^{-6}$
a 90	1 kW =	1·341	102	1000	1	0·239	0·9484
a 91	1 kcal/s =	5·614	426·9	4187	4·187	1	3·968
a 92	1 Btu/s =	1·415	107·6	1055	1·055	0·252	1

Other units

a 93 1 yard = 3 ft = 0·914 m

a 94 1 English mile = 1760 yds = 1609 m

a 95 1 Nautical mile = 1852 m

a 96 1 Geographical mile = 7420 m

a 97 1 long ton = 2240 lb = 1·016 Mg

a 98 1 short ton (US) = 2000 lb = 0·9072 Mg

a 99 1 long ton = 2240 lbf = 9·96 MN

a100 1 short ton (US) = 2000 lbf = 9·00 MN

a101 1 Imp. gallon (Imperial gallon) = 4·546 dm^3

a102 1 US gallon = 3·785 dm^3

a103 1 BTU/ft^3 = 9·547 kcal/m^3 = 39·964 kJ/m^3

a104 1 BTU/lb = 0·556 kcal/kg = 2·327 kJ/kg

a105 1 lbf/ft^2 = 4·882 kgf/m^2 = 47·8924 N/m^2

a106 1 lbf/in^2 (p.s.i.) = 0·0703 kgf/cm^2 = 0·6896 N/cm^2

a107 1 chain = 22 yds = 20·11 m

a108 1 Hundredweight (GB) (cwt) = 112 lbf = 498 kN

a109 1 Quarter (GB) = 28 lbf = 124·5 kN

a110 1 Stone (GB) = 14 lbf = 62·3kkN

AREAS

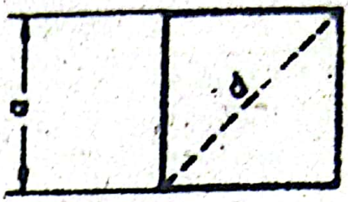
B 1

square

b 1 $A = a^2$

b 2 $a = \sqrt{A}$

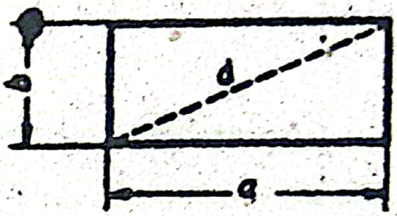
b 3 $d = a\sqrt{2}$



rectangle

b 4 $A = a\,b$

b 5 $d = \sqrt{a^2 + b^2}$

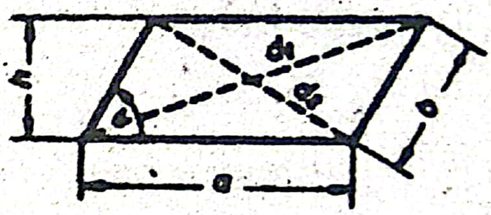


parallelogram

b 6 $A = a\,h = a\,b\,\sin\alpha$

b 7 $d_1 = \sqrt{(a + h\cot\alpha)^2 + h^2}$

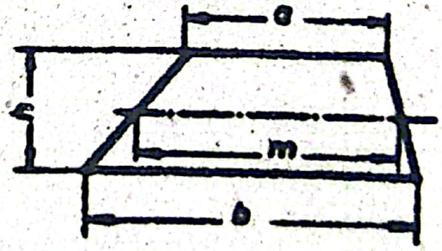
b 8 $d_2 = \sqrt{(a - h\cot\alpha)^2 + h^2}$



trapezium

b 9 $A = \frac{a + b}{2}h = m\,h$

b 10 $m = \frac{a + b}{2}$

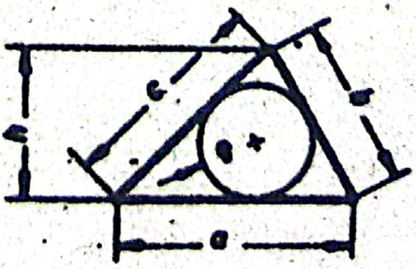


triangle

b 11 $A = \frac{a\,h}{2} = \rho\,s$

b 12 $= \sqrt{s(s-a)(s-b)(s-c)}$

b 13 $s = \frac{a + b + c}{2}$

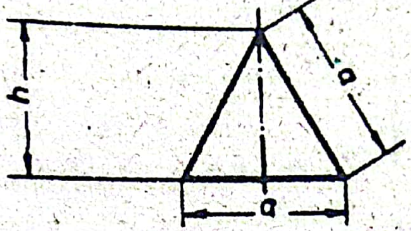


AREAS

equilateral triangle

b 14 $A = \frac{a^2}{4}\sqrt{3}$

b 15 $h = \frac{a}{2}\sqrt{3}$



pentagon

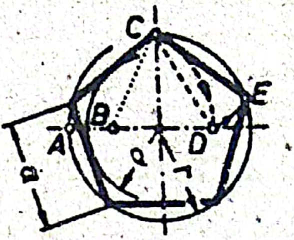
b 16 $A = \frac{5}{8}r^2\sqrt{10 + 2\sqrt{5}}$

b 17 $a = \frac{1}{2}r\sqrt{10 - 2\sqrt{5}}$

b 18 $\rho = \frac{1}{4}r\sqrt{6 + 2\sqrt{5}}$

construction:

$\overline{AB} = 0.5\,r, \quad \overline{BC} = \overline{BD}, \quad \overline{CD} = \overline{CE}$



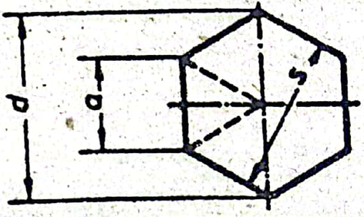
hexagon

b 19 $A = \frac{3}{2}a^2\sqrt{3}$

b 20 $d = 2a$

b 21 $= \frac{2}{\sqrt{3}}s = 1.155\,s$

b 22 $s = \frac{\sqrt{3}}{2}d = 0.866\,d$



octagon

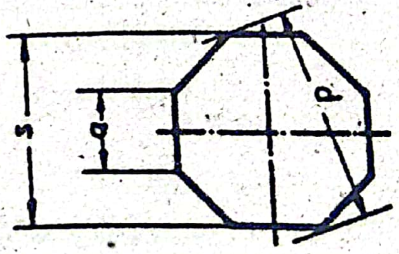
b 23 $A = 2as = 0.83\,s^2$

b 24 $= 2s\sqrt{d^2 - s^2}$

b 25 $a = s \times \tan 22.5^\circ = 0.415\,s$

b 26 $s = d \times \cos 22.5^\circ = 0.924\,d$

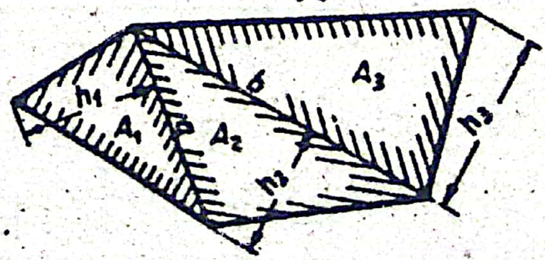
b 27 $d = \frac{s}{\cos 22.5^\circ} = 1.083\,s$



polygon

b 28 $A = A_1 + A_2 + A_3$

b 29 $= \frac{a h_1 + b h_2 + b h_3}{2}$



AREAS

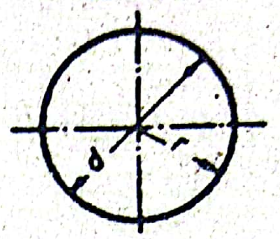
B 3

circle

b 30 $A = \frac{\pi}{4} d^2 = \pi r^2$

b 31 $\approx 0{\cdot}785\, d^2$

b 32 $U = 2 \pi r = \pi d$

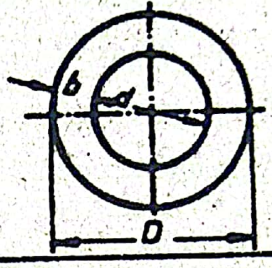


annulus

b 33 $A = \frac{\pi}{4} (D^2 - d^2)$

b 34 $= \pi (d + b) b$

b 35 $b = \frac{D - d}{2}$



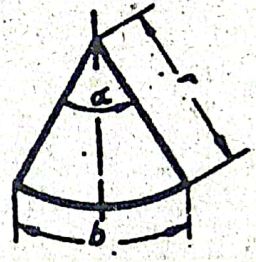
sector of a circle

b 36 $A = \frac{\pi}{360^\circ} r^2 \alpha = \frac{\hat{\alpha}}{2} r^2$

b 37 $= \frac{b\, r}{2}$

b 38 $b = \frac{\pi}{180^\circ} r \alpha$

b 39 $\hat{\alpha} = \frac{\pi}{180^\circ} \alpha$ ($\hat{\alpha} = \alpha$ in circular measure)



segment of a circle

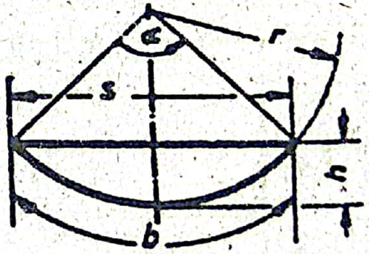
b 40 $s = 2\, r \sin\frac{\alpha}{2}$

b 41 $A = \frac{h}{6\, s}(3h^2 + 4s^2) = \frac{r^2}{2}(\hat{\alpha} - \sin\alpha)$

b 42 $r = \frac{h}{2} + \frac{s^2}{8h}$

b 43 $h = r(1 - \cos\frac{\alpha}{2}) = \frac{s}{2} \tan\frac{\alpha}{4}$

b 44 $\hat{\alpha}$ see formula b 39

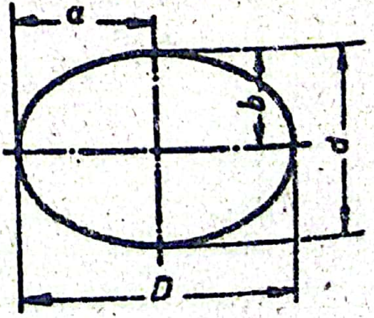


ellipse

b 45 $A = \frac{\pi}{4} D\, d = \pi\, a\, b$

b 46 $U \approx \pi \frac{D + d}{2}$

b 47 $= \pi(a+b)\left[1 + \frac{1}{4}\lambda^2 + \frac{1}{64}\lambda^4 + \frac{1}{256}\lambda^6 + \frac{25}{16384}\lambda^8 + \dots\right]$, where $\lambda = \frac{a-b}{a+b}$



SOLID BODIES

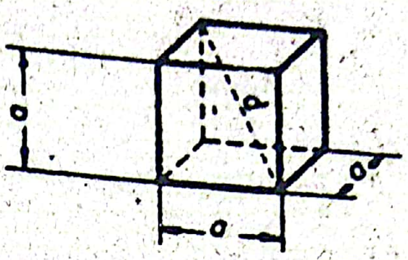
C 1

cube

c 1 $V = a^3$

c 2 $A_0 = 6\,a^2$

c 3 $d = \sqrt{3}\,a$

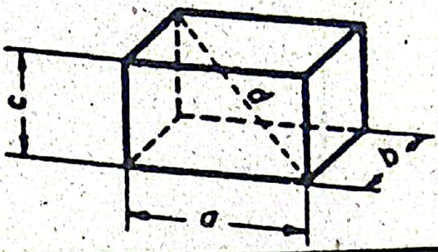


cuboid

c 4 $V = a\,b\,c$

c 5 $A_0 = 2(a\,b + a\,c + b\,c)$

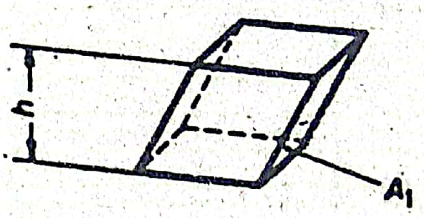
c 6 $d = \sqrt{a^2 + b^2 + c^2}$



parallelepiped

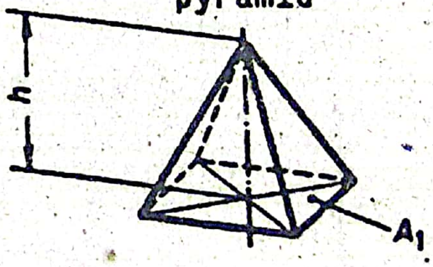
c 7 $V = A_1\,h$

(Cavalieri principle)



pyramid

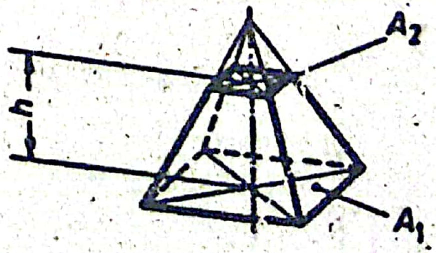
c 8 $V = \dfrac{A_1\,h}{3}$



frustum of pyramid

c 9 $V = \dfrac{h}{3}\left(A_1 + A_2 + \sqrt{A_1\,A_2}\right)$

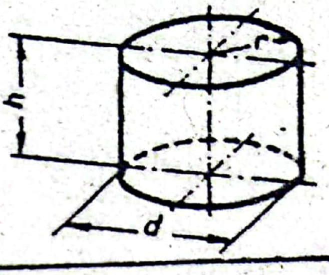
c10 $\approx h\,\dfrac{A_1 + A_2}{2}$



SOLID BODIES

C 2

cylinder

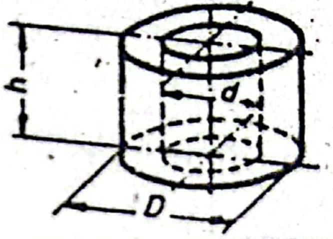


c 11 $V = \frac{\pi}{4} d^2 h$

c 12 $A_m = 2 \pi r h$

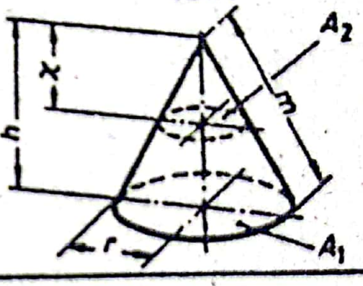
c 13 $A_o = 2 \pi r (r + h)$

hollow cylinder



c 14 $V = \frac{\pi}{4} h (D^2 - d^2)$

cone



c 15 $V = \frac{\pi}{3} r^2 h$

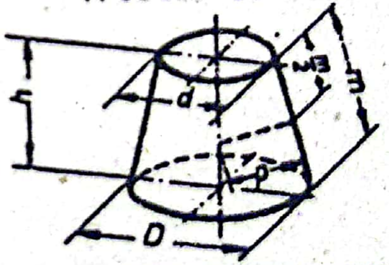
c 16 $A_m = \pi r m$

c 17 $A_o = \pi r (r + m)$

c 18 $m = \sqrt{h^2 + r^2}$

c 19 $A_2 : A_1 = x^2 : h^2$

frustum of cone

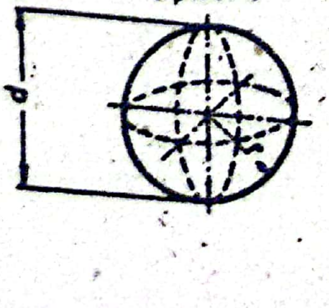


c 20 $V = \frac{\pi}{12} h (D^2 + Dd + d^2)$

c 21 $A_m' = \frac{\pi}{2} m (D + d) = 2 \pi p h$

c 22 $m = \sqrt{\left(\frac{D - d}{2}\right)^2 + h^2}$

sphere



c 23 $V = \frac{4}{3} \pi r^3 = \frac{1}{6} \pi d^3$

c. 24 $\approx 4 \cdot 189\, r^3$

c 25 $A_o = 4 \pi r^2 = \pi d^2$

SOLID BODIES

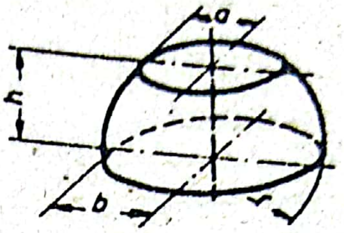
C 3

zone of a sphere

c 26 $V = \frac{\pi}{6} h (3a^2 + 3b^2 + h^2)$

c 27 $A_m = 2 \pi r h$

c 28 $A_o = \pi (2 r h + a^2 + b^2)$



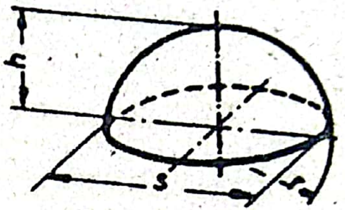
segment of a sphere

c 29 $V = \frac{\pi}{6} h \left(\frac{3}{4} s^2 + h^2 \right)$

$= \pi h^2 \left(r - \frac{h}{3} \right)$

c 30 $A_m = 2 \pi r h$

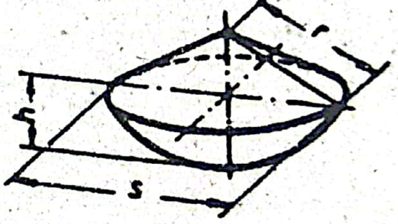
c 31 $= \frac{\pi}{4} (s^2 + 4 h^2)$



sector of a sphere

c 32 $V = \frac{2}{3} \pi r^2 h$

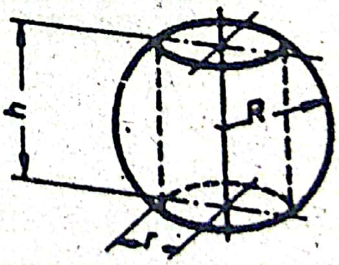
c 33 $A_o = \frac{\pi}{2} r (4 h + s)$



sphere with cylindrical boring

c 34 $V = \frac{\pi}{6} h^3$

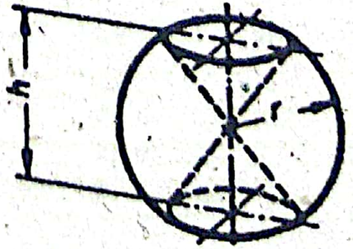
c 35 $A_o = 2 \pi h (R + r)$



sphere with conical boring

c 36 $V = \frac{2}{3} \pi r^2 h$

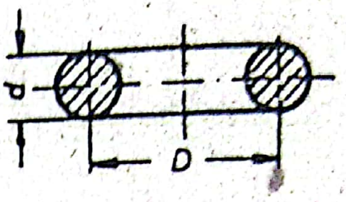
c 37 $A_o = 2 \pi r \left(h + \sqrt{r^2 - \frac{h^2}{4}} \right)$



torus

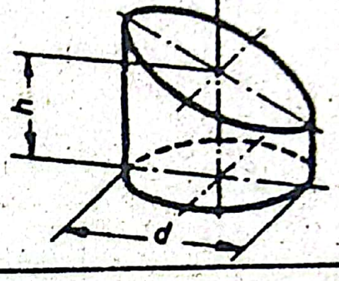
c 38 $V = \frac{\pi^2}{4} D d^2$

c 39 $A_o = \pi^2 D d$



sliced cylinder

c 40 $V = \frac{\pi}{4} d^2 h$

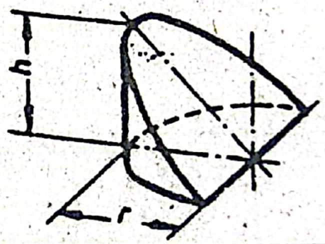


ungula

c 41 $V = \frac{2}{3} r^2 h$

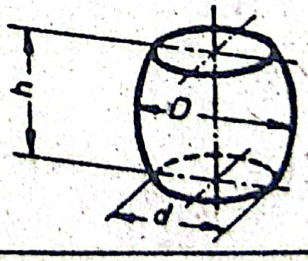
c 42 $A_m = 2 r h$

c 43 $A_o = A_m + \frac{\pi}{2} r^2 + \frac{\pi}{2} r \sqrt{r^2 + h^2}$



barrel

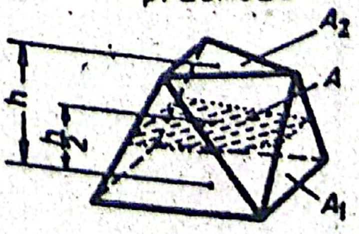
c 44 $V = \frac{\pi}{12} h (2 D^2 + d^2)$



prismoid

c 45 $V = \frac{h}{6} (A_1 + A_2 + 4 A)$

This formula may be used for calculations involving solids shown in fig. C1...C3 and thus spheres and parts of spheres.



ARITHMETIC

Powers, Roots

Rules for powers and roots

	general	numerical examples
d 1	$p\,a^n \pm q\,a^n = (p \pm q)a^n$	$3a^4 + 4a^4 = 7a^4$
d 2	$a^m\,a^n = a^{m+n}$	$a^8\,a^4 = a^{12}$
d 3	$\dfrac{a^m}{a^n} = a^{m-n}$	$\dfrac{a^8}{a^2} = a^{8-2} = a^6$
d 4	$(a^m)^n = (a^n)^m = a^{mn}$	$(a^3)^2 = (a^2)^3 = a^{2\times 3} = a^6$
d 5	$a^{-n} = \dfrac{1}{a^n}$	$a^{-4} = \dfrac{1}{a^4}$
d 6	$\dfrac{a^n}{b^n} = \left(\dfrac{a}{b}\right)^n$	$\dfrac{a^3}{b^3} = \left(\dfrac{a}{b}\right)^3$
d 7	$p\sqrt[n]{a} \pm q\sqrt[n]{a} = (p \pm q)\sqrt[n]{a}$	$4\sqrt[3]{x} + 7\sqrt[3]{x} = 11\sqrt[3]{x}$
d 8	$\sqrt[n]{a\,b} = \sqrt[n]{a}\,\sqrt[n]{b}$	$\sqrt[4]{16\times 81} = \sqrt[4]{16}\times\sqrt[4]{81}$
d 9	$\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}} = \left(\dfrac{a}{b}\right)^{\frac{1}{n}}$	$\dfrac{\sqrt{8}}{\sqrt{2}} = \sqrt{4} = 2$
d10	$\sqrt[nx]{a^{mx}} = \sqrt[n]{a^m}$	$\sqrt[6]{a^8} = \sqrt[3]{a^4}$
d11	$\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m = a^{\frac{m}{n}}$ +)	$\sqrt[4]{a^3} = \left(\sqrt[4]{a}\right)^3 = a^{\frac{3}{4}}$
d12	$\sqrt{-a} = i\sqrt{a}$	$\sqrt{-9} = i\sqrt{9} = i\,3$

+) Not applicable to special calculations } e.g. $\sqrt{(-2)^2} = \pm 2$ or $(\sqrt{-2})^2 = -2$

Note: Exponents of powers and roots have to be non-dimensional quantities!

ARITHMETIC

Logarithms

D 2

General

	system	log to the base of	terminology
d 13	$\log_a$	a	log to base a
d 14	$\log_{10} = \lg$	10	common log
d 15	$\log_e = \ln$	e	natural log
d 16	$\log_2 = \mathrm{ld}$	2	log to base 2

The symbols in $\log_a x = b$ are called: a base
x antilogarithm
b logarithm(log)

Rules for logarithmic calculations

d 17 $\log_a (x\,y) = \log_a x + \log_a y$

d 18 $\log_a \dfrac{x}{y} = \log_a x - \log_a y$

d 19 $\log_a x^n = n \times \log_a x$

d 20 $\log_a \sqrt[n]{x} = \dfrac{1}{n} \log_a x$

Exponential equation

d 21 $a^x = b = e^{x \ln a}$

d 22 whence: $x = \dfrac{\log b}{\log a}$ | $a = \sqrt[x]{b}$

Conversion of logarithms

d 23 $\lg x = \lg e \times \ln x = 0{\cdot}434294 \times \ln x$

d 24 $\ln x = \dfrac{\lg x}{\lg e} = 2{\cdot}302585 \times \lg x$

d 25 Base of the natural logs $e = 2{\cdot}718\,281\,83\ldots$

Key to common logarithm of a number

d 26 $\lg 0{\cdot}01 = -2.$ or $8.\ldots-10$
d 27 $\lg 0{\cdot}1 = -1.$ or $9.\ldots-10$
d 28 $\lg 1 = 0.$
d 29 $\lg 10 = 1.$
d 30 $\lg 100 = 2.$
etc.

Note: The antilogarithm always has to be a non-dimensional quantity

ARITHMETIC

Powers, Roots

Conversion of general algebraic expressions

d 31 $(a \pm b)^2 = a^2 \pm 2ab + b^2$

d 32 $(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$

d 33 $$(a + b)^n = a^n + \frac{n}{1} a^{n-1} b + \frac{n(n-1)}{1 \times 2} a^{n-2} b^2 + \frac{n(n-1)(n-2)}{1 \times 2 \times 3} a^{n-3} b^3 + \dots b^n$$

d 34 $(a + b + c)^2 = a^2 + 2ab + 2ac + b^2 + 2bc + c^2$

d 35 $(a - b + c)^2 = a^2 - 2ab + 2ac + b^2 - 2bc + c^2$

d 36 $a^2 - b^2 = (a + b)(a - b)$

d 37 $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

d 38 $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

d 39 $$a^n - b^n = (a - b)(a^{n-1} + a^{n-2} b + a^{n-3} b^2 + \dots + ab^{n-2} + b^{n-1})$$

Quadratic equation (equation of the second degree)

d 40 Normal form $ax^2 + bx + c = 0$

d 41 Solutions $x_1, x_2 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Arithmetical calculation of a square root

	example	explanation, ()-values corresponding to example
d 43	$\begin{array}{l} \sqrt{21\;43{\cdot}69} = 46{\cdot}3 \\ \underline{16} \\ \;\;5\;43 \quad : \quad 86 \\ \;\;\underline{5\;16} \\ \quad 27\;69 \;:\; 923 \\ \quad \underline{27\;69} \\ \quad --\;\;-- \end{array}$	a) Subdivide antilog into groups of 2 figures starting at the decimal point. b) Calculate greatest square root contained in 1st group(21).Square resulting integer (4), which is 1st figure of final result, and subtract square(16) from 1st group.

c) Add next group (43) to remainder (5) and bring down double the result obtained hitherto (2 × 4 = 8). Add result (6),which is next figure of final result,to doubled previous result. Multiply this divison(86)by latest figure of final result(6). Calculate difference(27). Repeat until required accuracy is obtained

ARITHMETIC

Binomial theorem

D 4

Binomial theorem

d 44 $$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}\,b + \binom{n}{2}a^{n-2}\,b^2 + \binom{n}{3}a^{n-3}\,b^3 + \ldots$$

n must be a whole number

d 45 $$(a+b)^4 = 1a^4 + \frac{4}{1}a^{4-1}\,b + \frac{4\times3}{1\times2}a^{4-2}\,b^2 + \frac{4\times3\times2}{1\times2\times3}a^{4-3}\,b^3 + b^4$$

$$= a^4 + 4a^3\,b + 6a^2\,b^2 + 4a\,b^3 + b^4$$

Diagrammatic solution

Coefficient - Pascal triangle

d 46 $$(a+b)^0 \qquad 1$$

d 47 $$(a+b)^1 \qquad 1 \quad 1$$

d 48 $$(a+b)^2 \qquad 1 \quad 2 \quad 1$$

d 49 $$(a+b)^3 \qquad 1 \quad 3 \quad 3 \quad 1$$

d 50 $$(a+b)^4 \qquad 1 \quad 4 \quad \underbrace{6 \quad 4} \quad 1$$

d 51 $$(a+b)^5 \qquad 1 \quad \underbrace{5 \quad 10} \quad 10 \quad 5 \quad 1$$

d 52 $$(a+b)^6 \qquad 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$

Continue with each line starting and finishing with 1. The second and penultimate numbers should be the exponents, the others the sum of those to the right and left immediately above them.

Exponents

The sum of the exponents a and b in each separate term is equal to the binomial exponent n. As the power of a decreases the power of b increases.

Signs

$(a+b)$ is always positive
$(a-b)$ is initially positive and changes from term to term

Examples

d 53 $$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

d 54 $$(a-b)^5 = +a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$$

ARITHMETIC

Determinants and linear equations

D 5

Second order determinants

d 55
$$\begin{aligned} a_{11}\,x + a_{12}\,y &= r_1\\ a_{21}\,x + a_{22}\,y &= r_2 \end{aligned} \qquad D = \begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22} \end{vmatrix} = a_{11}\,a_{22} - a_{21}\,a_{12}$$

insert r column in place of

column x	column y
$D_1 = \begin{vmatrix} r_1 & a_{12}\\ r_2 & a_{22} \end{vmatrix} = r_1\,a_{22} - r_2\,a_{12}$	$D_2 = \begin{vmatrix} a_{11} & r_1\\ a_{21} & r_2 \end{vmatrix} = r_2\,a_{11} - r_1\,a_{21}$
$x = \dfrac{D_1}{D}$	$y = \dfrac{D_2}{D}$

d 56

Third order determinants (Sarrus rule)

d 57
$$\begin{aligned} a_{11}\,x + a_{12}\,y + a_{13}\,z &= r_1\\ a_{21}\,x + a_{22}\,y + a_{23}\,z &= r_2\\ a_{31}\,x + a_{32}\,y + a_{33}\,z &= r_3 \end{aligned}$$

d 58
$$D = \left|\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12}\\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22}\\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right| \begin{aligned} = {} & a_{11}\,a_{22}\,a_{33} + a_{12}\,a_{23}\,a_{31}\\ & + a_{13}\,a_{21}\,a_{32} - a_{13}\,a_{22}\,a_{31}\\ & - a_{11}\,a_{23}\,a_{32} - a_{12}\,a_{21}\,a_{33} \end{aligned}$$

insert r column for x column:

d 59
$$D_1 = \left|\begin{array}{ccc|cc} r_1 & a_{12} & a_{13} & r_1 & a_{12}\\ r_2 & a_{22} & a_{23} & r_2 & a_{22}\\ r_3 & a_{32} & a_{33} & r_3 & a_{32} \end{array}\right| \begin{aligned} = {} & r_1\,a_{22}\,a_{33} + a_{12}\,a_{23}\,r_3\\ & + a_{13}\,r_2\,a_{32} - a_{13}\,a_{22}\,r_3\\ & - r_1\,a_{23}\,a_{32} - a_{12}\,r_2\,a_{33} \end{aligned}$$

determine D_2 and D_3 similarly by replacing the y- and z-column by the r-column:

d 60
$$x = \frac{D_1}{D} \qquad y = \frac{D_2}{D} \qquad z = \frac{D_3}{D}$$

continued on D 6

Determinants of more than the 2nd order:

(The Sarrus Rule, see D 5, may be used for determinants of higher order than the 3rd).

By adding or substracting suitable multiples of two rows or columns, endeavour to obtain zero values. Expand the determinant starting from the row or column containing most zeros. Alternate the signs of terms, starting with a_{11} as +.

Example:

d 61
$$\begin{array}{cccc} \overset{+}{a_{11}} & \overset{-}{a_{12}} & \overset{+}{a_{13}} & \overset{-}{0} \\ a_{21} & a_{22} & a_{23} & \overset{+}{a_{24}} \\ a_{31} & a_{32} & a_{33} & \overset{-}{a_{34}} \\ a_{41} & a_{42} & a_{43} & \overset{+}{0} \end{array}$$

Expand on 4th column:

d 62
$$a_{24}\begin{vmatrix} \overset{+}{a_{11}} & \overset{-}{a_{12}} & \overset{+}{a_{13}} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} \quad -a_{34}\begin{vmatrix} \overset{+}{a_{11}} & \overset{-}{a_{12}} & \overset{+}{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

Further expand as:

d 63
$$D = a_{24}\left(a_{11}\begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} - a_{12}\begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix} + a_{13}\begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}\right) - a_{34}\left(\cdot\ \cdot\ \cdot\right)$$

To form the determinants D_1, D_2, ... (see D 5) substitute the r column for the first, second, ... column of D, and evaluate in the same way as for D.

For determinant of the nth order, find $u_{1 \ldots n}$ from the formulae:

d 64
$$u_1 = \frac{D_1}{D}, \quad u_2 = \frac{D_2}{D}, \quad \ldots u_n = \frac{D_n}{D}$$

Note: For determinants of the nth order continue until determinants of the 3rd order have been obtained.

ARITHMETIC

Series

D 7

Arithmetic series

The series 1, 4, 7, 10 etc. is called an arithmetic series (the difference between two consecutive terms is constant).

Formulae:

d 65 $$a_n = a_1 + (n-1)\,d$$

d 66 $$s_n = \frac{n}{2}(a_1 + a_n) = a_1 n + \frac{n(n-1)\,d}{2}$$

Arithmetic mean:

Each term of an arithmetic series is the arithmetic mean a_m of its adjacent terms a_{m-1} and a_{m+1}.

Thus, the mth term is

d 67 $$a_m = \frac{a_{m-1} + a_{m+1}}{2} \qquad \text{for } 1 < m < n$$

(e.g. in the above series $a_3 = \frac{4 + 10}{2} = 7$)

Geometric series

The series 1, 2, 4, 8, etc. is called a geometric series (the quotient of two consecutive terms is constant).

Formulae:

d 68 $$a_n = a_1\,q^{n-1}$$

d 69 $$s_n = a_1\,\frac{q^n - 1}{q - 1} = \frac{q \cdot a_n - a_1}{q - 1}$$

Geometric mean:

Each term of a geometric series is the geometric mean a_m of its adjacent terms a_{m-1} and a_{m+1}.

Thus, the mth term is

d 70 $$a_m = \sqrt{a_{m-1}\,a_{m+1}} \qquad \text{for } 1 < m < n$$

(e.g. in the above series $a_3 = \sqrt{2 \times 8} = 4$)

For infinite geometric series ($n \to \infty$; $|q| < 1$) the following statements apply

d 71 $$a_n = \lim_{n\to\infty} a_n = 0\,; \qquad s_n = \lim_{n\to\infty} s_n = a_1\,\frac{1}{1-q}$$

a_1 : initial term
a_n : final term
d : difference between two consecutive terms

n : number of terms
s_n : sum of all terms
q : quotient of two consecutive terms.

Binomial series

d 72 $f(x) = (1 \pm x)^{a} = 1 \pm \binom{a}{1}x + \binom{a}{2}x^{2} \pm \binom{a}{3}x^{3} + \dots$

a may be either positive or negative, a whole number or a fraction.

Expansion of the binomial coefficient:

$$\binom{a}{n} = \frac{a(a-1)(a-2)(a-3)\dots(a-n+1)}{1 \times 2 \times 3 \dots \times n}$$

Examples:

		for
d 73	$\frac{1}{1 \pm x} = (1 \pm x)^{-1} = 1 \mp x + x^{2} \mp x^{3} + \dots$	$\lvert x \rvert < 1$
d 74	$\sqrt{1 \pm x} = (1 \pm x)^{\frac{1}{2}} = 1 \pm \frac{1}{2}x - \frac{1}{8}x^{2} \pm \frac{1}{16}x^{3} - \dots$	$\lvert x \rvert < 1$
d 75	$\frac{1}{\sqrt{1 \pm x}} = (1 \pm x)^{-\frac{1}{2}} = 1 \mp \frac{1}{2}x + \frac{3}{8}x^{2} \mp \frac{5}{16}x^{3} + \dots$	$\lvert x \rvert < 1$

Taylor series

d 76 $f(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^{2} + \dots$

putting $a = 0$ gives the MacLaurin series:

d 77 $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^{2} + \dots$

Examples:

		for
d 78	$e^{x} = 1 + \frac{x}{1!} + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \dots$	all x
d 79	$a^{x} = 1 + \frac{x \ln a}{1!} + \frac{(x \ln a)^{2}}{2!} + \frac{(x \ln a)^{3}}{3!} + \dots$	all x
d 80	$\ln x = 2\left[\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^{3} + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^{5} + \dots\right]$	$x > 0$
d 81	$\ln(1+x) = x - \frac{x^{2}}{2} + \frac{x^{3}}{3} - \frac{x^{4}}{4} + \frac{x^{5}}{5} - \dots$	$-1 < x \le +1$
d 82	$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$	

continued on D 9

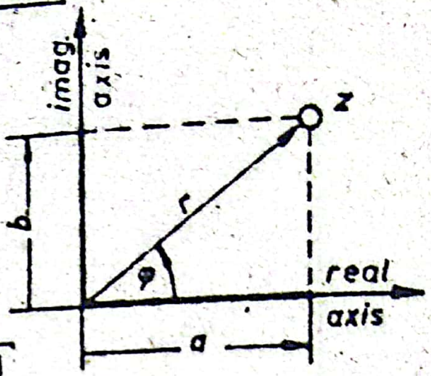
Taylor series
(continued)

	Examples	for
d 83	$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$	all x
d 84	$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$	all x
d 85	$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \dots$	$\lvert x\rvert < \frac{\pi}{2}$
d 86	$\cot x = \frac{1}{x} - \frac{1}{3}x - \frac{1}{45}x^3 - \frac{2}{945}x^5 - \dots$	$0 < \lvert x\rvert$ $\lvert x\rvert < \pi$
d 87	$\arcsin x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\times3}{2\times4}\frac{x^5}{5} + \frac{1\times3\times5}{2\times4\times6}\frac{x^7}{7} + \dots$	$\lvert x\rvert \leqq 1$
d 88	$\arccos x = \frac{\pi}{2} - \arcsin x$	$\lvert x\rvert \leqq 1$
d 89	$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots$	$\lvert x\rvert \leqq 1$
d 90	$\operatorname{arccot} x = \frac{\pi}{2} - \arctan x$	$\lvert x\rvert \leqq 1$
d 91	$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \dots$	all x
d 92	$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \dots$	all x
d 93	$\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \dots$	$\lvert x\rvert < \frac{\pi}{2}$
d 94	$\coth x = \frac{1}{x} + \frac{1}{3}x - \frac{1}{45}x^3 + \frac{2}{945}x^5 - \dots$	$0 < \lvert x\rvert$ $\lvert x\rvert < \pi$
d 95	$\operatorname{arsinh} x = x - \frac{1}{2}\frac{x^3}{3} + \frac{1\times3}{2\times4}\frac{x^5}{5} - \frac{1\times3\times5}{2\times4\times6}\frac{x^7}{7} + \dots$	$\lvert x\rvert < 1$
d 96	$\operatorname{arcosh} x = \ln 2x - \frac{1}{2}\frac{1}{2x^2} - \frac{1\times3}{2\times4}\frac{1}{4x^4} - \frac{1\times3\times5}{2\times4\times6}\frac{1}{6x^6} + \dots$	$\lvert x\rvert > 1$
d 97	$\operatorname{artanh} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \dots$	$\lvert x\rvert < 1$
d 98	$\operatorname{arcoth} x = \frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \frac{1}{7x^7} + \dots$	$\lvert x\rvert < 1$

Complex numbers

General

$z = r e^{i\varphi} = a + i b$
a = real part of z
b = imaginary part of z
r = absolute value of z
= or modulus of z
φ = argument of z
a and b are real



$i = \sqrt{-1}$

d 99 $i^1 = +i$ $i^{-1} = -i$
d100 $i^2 = -1$ $i^{-2} = -1$
d101 $i^3 = -i$ $i^{-3} = +i$
d102 $i^4 = +1$ $i^{-4} = +1$
d103 $i^5 = +i$ $i^{-5} = -i$
d104 etc.

Note: In electrical engineering the letter j is used for i to avoid confusion.

In the Cartesian coordinate system:

d105 $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$

d106 $z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$

d107 $z_1 \times z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$

d108 $\dfrac{z_1}{z_2} = \dfrac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i\dfrac{-a_1 b_2 + a_2 b_1}{a_2^2 + b_2^2}$

d109 $a^2 + b^2 = (a + ib)(a - ib)$

d110 $\sqrt{a \pm ib} = \sqrt{\dfrac{a + \sqrt{a^2 + b^2}}{2}} \pm i\sqrt{\dfrac{-a + \sqrt{a^2 + b^2}}{2}}$

d111 Where $a_1 = a_2$ and $b_1 = b_2$, then $z_1 = z_2$

continued on D 11

Complex numbers (continued)

In the polar coordinate system:

d 112 $z = r(\cos\varphi + i\sin\varphi) = a + ib$

d 113 $r = +\sqrt{a^2 + b^2}$

d 114 $\varphi = \arctan\frac{b}{a}$

d 115 $\sin\varphi = \frac{b}{r} \quad \Big| \quad \cos\varphi = \frac{a}{r} \quad \Big| \quad \tan\varphi = \frac{b}{a}$

d 116 $z_1 \times z_2 = r_1 \times r_2\left[\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)\right]$

d 117 $\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)\right] \quad (z_2 \neq 0)$

d 118 $z^n = r^n\left[\cos(n\varphi) + i\sin(n\varphi)\right]$ ($n > 0$ integral)

d 119 $\sqrt[n]{z} = \left|\sqrt[n]{r}\right|\left(\cos\frac{\varphi + 2\pi k}{n} + i\sin\frac{\varphi + 2\pi k}{n}\right)$

d 120 $\sqrt[n]{1} = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$ (nth roots of unity)

in formula d 119 and d 120 $k = 0, 1, 2, \ldots, n-1$

d 121 $e^{i\hat{\varphi}} = \cos\varphi + i\sin\varphi$

d 122 $e^{-i\hat{\varphi}} = \cos\varphi - i\sin\varphi = \frac{1}{\cos\varphi + i\sin\varphi}$

d 123 $\left|e^{-i\hat{\varphi}}\right| = \sqrt{\cos^2\varphi + \sin^2\varphi} = 1$

d 124 $\cos\varphi = \frac{e^{i\hat{\varphi}} + e^{-i\hat{\varphi}}}{2} \quad \Big| \quad \sin\varphi = \frac{e^{i\hat{\varphi}} - e^{-i\hat{\varphi}}}{2i}$

d 125 $\ln z = \ln r + i(\hat{\varphi} + 2\pi k) \qquad (k = 0, \pm 1, \pm 2, \ldots)$

Where $r_1 = r_2$ and $\varphi_1 = \varphi_2 + 2\pi k$, then $z_1 = z_2$

Note: φ should be measured along the arc
k is any arbitrary whole number

Compound interest calculation

d 126 $$k_n = k_0 \, q^n$$

d 127 $$n = \frac{\lg \dfrac{k_n}{k_0}}{\lg q}$$

Annuity interest calculation

d 128 $$k_n = k_0 \, q^n - r \, \frac{q^n - 1}{q - 1}$$

d 129 $$r = \frac{(k_0 \, q^n - k_n)(q - 1)}{q^n - 1}$$

d 130 $$n = \frac{\lg \dfrac{r - k_n(q - 1)}{r - k_0(q - 1)}}{\lg q}$$

Where $k_n = 0$, we get the "redemption formulae"

Deposit calculation
(savings bank formula)

d 131 $$k_n = k_0 \, q^n + r \, \frac{q^n - 1}{q - 1}$$

d 132 $$r = \frac{(k_n - k_0 \, q^n)(q - 1)}{q^n - 1}$$

d 133 $$n = \frac{\lg \dfrac{k_n(q - 1) + r}{k_0(q - 1) + r}}{\lg q}$$

Let

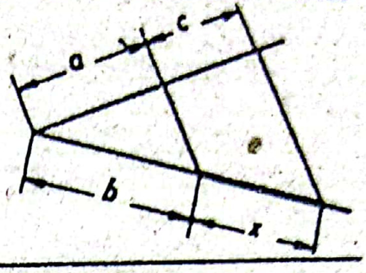
k_0: initial capital
k_n: capital after n years
r : annual pension (withdrawals)

n : number of years
q : 1 + p
p : rate of interest (e.g. 0·[illegible])

d 134 $x = \frac{b\,c}{a}$

d 135 $a : b = c : x$

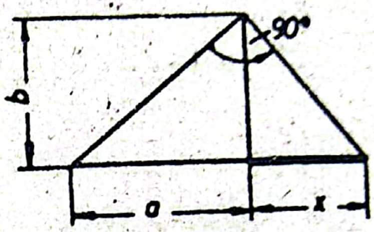
x : 4th proportional



d 136 $x = \frac{b^2}{a}$

d 137 $a : b = b : x$

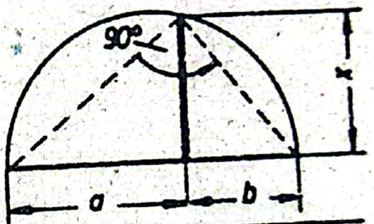
x : 3rd proportional



d 138 $x = \sqrt{a\,b}$

d 139 $a : x = x : b$

x : mean proportional

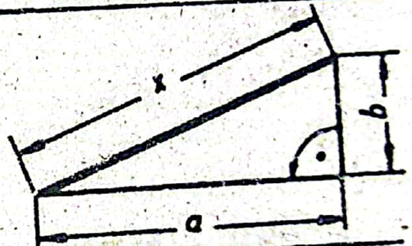


d 140 $x^2 = a^2 + b^2$

or

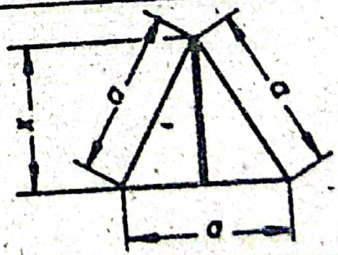
d 141 $x = \sqrt{a^2 + b^2}$

x : hypothenuse of a right-angled triangle



d 142 $x = \frac{a}{2}\sqrt{3}$

x : height of an equilateral triangle

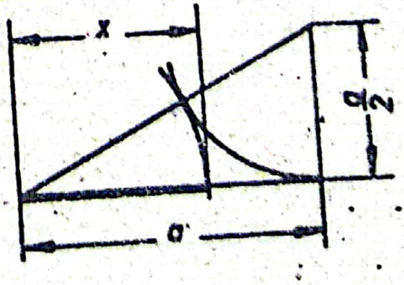


d 143 $x = \frac{a}{2}(\sqrt{5} - 1)$

$\approx a\ 0{\cdot}618$

$a : x = x : (a-x)$

x : larger section of a repeatedly subdivided line (golden section)



FUNCTIONS OF A CIRCLE

Basic terms

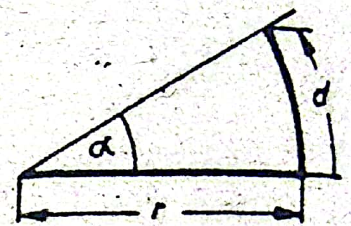
E 1

Circular and angular measure of a plane angle

Circular measure

Circular measure is the ratio of the distance d measured along the arc to the radius r.

It is given the unit "radian" which has no dimensions.



e 1 $$\alpha = \frac{d}{r} \text{ (rad)}$$

Unit: rad

Angular measure

Angular measure is obtained by dividing the angle subtended at the centre of a circle into 360 equal divisions known as "degrees".

Unit: °

e 2 A degree is divided into 60 minutes (unit: '),
e 3 a minute is divided into 60 seconds (unit: ").

Relation between circular and angular measure

By considering a circle, it may be seen that

e 4 $$360^\circ = 2\pi \text{ radians}$$

e 5 or $$1 \text{ rad} = 57{\cdot}2958^\circ$$

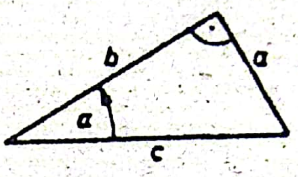
e 6

degrees	0°	30°	45°	60°	75°	90°	180°	270°	360°
radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5}{12}\pi$	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$	2π
	0	0·52	0·79	1·05	1·31	1·57	3·14	4·71	6·28

FUNCTIONS OF A CIRCLE

General terms

Right angled triangle



e 7 $\sin\alpha = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{a}{c}$

e 8 $\cos\alpha = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{b}{c}$

e 9 $\tan\alpha = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{a}{b}$ | $\cot\alpha = \dfrac{\text{adjacent}}{\text{opposite}} = \dfrac{b}{a}$

Functions of the more important angles

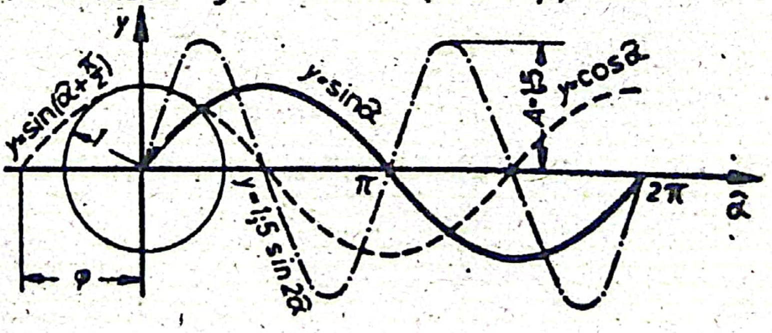
e 10

angle α	0°	30°	45°	60°	75°	90°	180°	270°	360°
sin α	0	0·500	0·707	0·866	0·966	1	0	−1	0
cos α	1	0·866	0·707	0·500	0·259	0	−1	0	1
tan α	0	0·577	1·000	1·732	3·732	∞	0	∞	0
cot α	∞	1·732	1·000	0·577	0·268	0	∞	0	∞

Relations between sine and cosine functions

Basic equations

e 11 Sine function $y = A\sin(k\alpha - \varphi)$
e 12 Cosine function $y = A\cos(k\alpha - \varphi)$



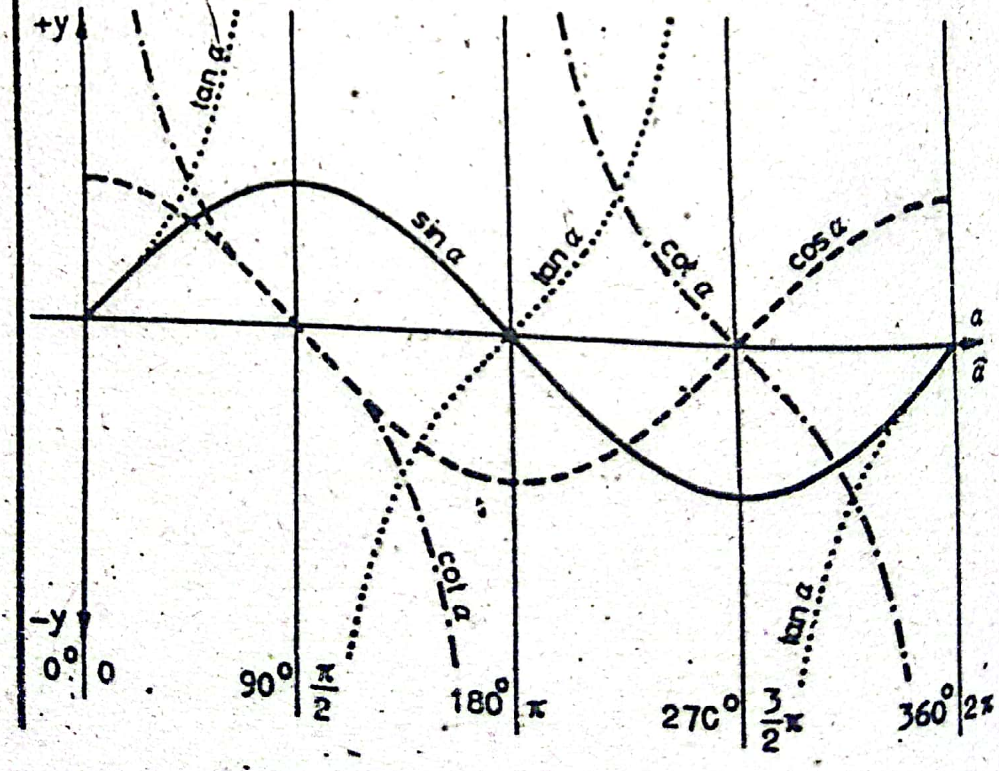
——	sine curve	with an amplitude of	A = 1 and	k = 1
—·—	sine curve		A = 1·5 and	k = 2
----	cosine curve		A = 1 and	k = 1

or sine curve with a phase shift of $\varphi = -\dfrac{\pi}{2}$

FUNCTIONS OF A CIRCLE

Quadrants

e 15	$\sin(90^\circ - \alpha)$	$= +\cos\alpha$	$\sin(90^\circ + \alpha)$	$= +\cos\alpha$
e 16	$\cos($ " $)$	$= +\sin\alpha$	$\cos($ " $)$	$= -\sin\alpha$
e 17	$\tan($ " $)$	$= +\cot\alpha$	$\tan($ " $)$	$= -\cot\alpha$
e 18	$\cot($ " $)$	$= +\tan\alpha$	$\cot($ " $)$	$= -\tan\alpha$
e 19	$\sin(180^\circ - \alpha)$	$= +\sin\alpha$	$\sin(180^\circ + \alpha)$	$= -\sin\alpha$
e 20	$\cos($ " $)$	$= -\cos\alpha$	$\cos($ " $)$	$= -\cos\alpha$
e 21	$\tan($ " $)$	$= -\tan\alpha$	$\tan($ " $)$	$= +\tan\alpha$
e 22	$\cot($ " $)$	$= -\cot\alpha$	$\cot($ " $)$	$= +\cot\alpha$
e 23	$\sin(270^\circ - \alpha)$	$= -\cos\alpha$	$\sin(270^\circ + \alpha)$	$= -\cos\alpha$
e 24	$\cos($ " $)$	$= -\sin\alpha$	$\cos($ " $)$	$= +\sin\alpha$
e 25	$\tan($ " $)$	$= +\cot\alpha$	$\tan($ " $)$	$= -\cot\alpha$
e 26	$\cot($ " $)$	$= +\tan\alpha$	$\cot($ " $)$	$= -\tan\alpha$
e 27	$\sin(360^\circ - \alpha)$	$= -\sin\alpha$	$\sin(360^\circ + \alpha)$	$= +\sin\alpha$
e 28	$\cos($ " $)$	$= +\cos\alpha$	$\cos($ " $)$	$= +\cos\alpha$
e 29	$\tan($ " $)$	$= -\tan\alpha$	$\tan($ " $)$	$= +\tan\alpha$
e 30	$\cot($ " $)$	$= -\cot\alpha$	$\cot($ " $)$	$= +\cot\alpha$
e 31	$\sin(-\alpha)$	$= -\sin\alpha$	$\sin(\alpha \pm n \times 360^\circ)$	$= +\sin\alpha$
e 32	$\cos($ " $)$	$= +\cos\alpha$	$\cos($ " $)$	$= +\cos\alpha$
e 33	$\tan($ " $)$	$= -\tan\alpha$	$\tan(\alpha \pm n \times 180^\circ)$	$= +\tan\alpha$
e 34	$\cot($ " $)$	$= -\cot\alpha$	$\cot($ " $)$	$= +\cot\alpha$



FUNCTIONS OF A CIRCLE

Trigonometric conversions

E4

Basic identities

e 35 $\sin^2\alpha + \cos^2\alpha = 1$ $\quad\quad$ $\tan\alpha \times \cot\alpha = 1$

e 36 $1 + \tan^2\alpha = \dfrac{1}{\cos^2\alpha} = \sec^2\alpha$ $\quad\quad$ $1 + \cot^2\alpha = \dfrac{1}{\sin^2\alpha} = \operatorname{cosec}^2\alpha$

Sum and difference of angles

e 37 $\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta$

e 38 $\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$

e 39 $\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha \times \tan\beta}$

e 40 $\cot(\alpha \pm \beta) = \dfrac{\cot\alpha \times \cot\beta \mp 1}{\cot\beta \pm \cot\alpha}$

Sum and difference of functions of angles

e 41 $\sin\alpha + \sin\beta = 2\sin\dfrac{\alpha + \beta}{2}\cos\dfrac{\alpha - \beta}{2}$

e 42 $\sin\alpha - \sin\beta = 2\cos\dfrac{\alpha + \beta}{2}\sin\dfrac{\alpha - \beta}{2}$

e 43 $\cos\alpha + \cos\beta = 2\cos\dfrac{\alpha + \beta}{2}\cos\dfrac{\alpha - \beta}{2}$

e 44 $\cos\alpha - \cos\beta = -2\sin\dfrac{\alpha + \beta}{2}\sin\dfrac{\alpha - \beta}{2}$

e 45 $\tan\alpha \pm \tan\beta = \dfrac{\sin(\alpha \pm \beta)}{\cos\alpha \times \cos\beta}$

e 46 $\cot\alpha \pm \cot\beta = \dfrac{\sin(\beta \pm \alpha)}{\sin\alpha \times \sin\beta}$

e 47 $\sin\alpha \cos\beta = \dfrac{1}{2}\sin(\alpha + \beta) + \dfrac{1}{2}\sin(\alpha - \beta)$

e 48 $\cos\alpha \cos\beta = \dfrac{1}{2}\cos(\alpha + \beta) + \dfrac{1}{2}\cos(\alpha - \beta)$

e 49 $\sin\alpha \sin\beta = \dfrac{1}{2}\cos(\alpha - \beta) - \dfrac{1}{2}\cos(\alpha + \beta)$

e 50 $\tan\alpha \tan\beta = \dfrac{\tan\alpha + \tan\beta}{\cot\alpha + \cot\beta} = -\dfrac{\tan\alpha - \tan\beta}{\cot\alpha - \cot\beta}$

e 51 $\cot\alpha \cot\beta = \dfrac{\cot\alpha + \cot\beta}{\tan\alpha + \tan\beta} = -\dfrac{\cot\alpha - \cot\beta}{\tan\alpha - \tan\beta}$

e 52 $\cot\alpha \tan\beta = \dfrac{\cot\alpha + \tan\beta}{\tan\alpha + \cot\beta} = -\dfrac{\cot\alpha - \tan\beta}{\tan\alpha - \cot\beta}$

FUNCTIONS OF A CIRCLE

Trigonometric conversions

Ratios between simple, double and half angles

	$\sin\alpha =$	$\cos\alpha =$	$\tan\alpha =$	$\cot\alpha =$
e 53	$\cos(90^\circ-\alpha)$	$\sin(90^\circ-\alpha)$	$\cot(90^\circ-\alpha)$	$\tan(90^\circ-\alpha)$
e 54	$\sqrt{1-\cos^2\alpha}$	$\sqrt{1-\sin^2\alpha}$	$\frac{1}{\cot\alpha}$	$\frac{1}{\tan\alpha}$
e 55	$2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$	$\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}$	$\frac{\sin\alpha}{\cos\alpha}$	$\frac{\cos\alpha}{\sin\alpha}$
e 56	$\frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}}$	$\frac{\cot\alpha}{\sqrt{1+\cot^2\alpha}}$	$\frac{\sin\alpha}{\sqrt{1-\sin^2\alpha}}$	$\frac{\cos\alpha}{\sqrt{1-\cos^2\alpha}}$
e 57	$\sqrt{\cos^2\alpha-\cos(2\alpha)}$	$1-2\sin^2\frac{\alpha}{2}$	$\sqrt{\frac{1}{\cos^2\alpha}-1}$	$\sqrt{\frac{1}{\sin^2\alpha}-1}$
e 58	$\sqrt{\frac{1-\cos(2\alpha)}{2}}$	$\sqrt{\frac{1+\cos(2\alpha)}{2}}$	$\frac{\sqrt{1-\cos^2\alpha}}{\cos\alpha}$	$\frac{\sqrt{1-\sin^2\alpha}}{\sin\alpha}$
e 59	$\frac{1}{\sqrt{1+\cot^2\alpha}}$	$\frac{1}{\sqrt{1+\tan^2\alpha}}$		
e 60	$\frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$	$\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$	$\frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}$	$\frac{\cot^2\frac{\alpha}{2}-1}{2\cot\frac{\alpha}{2}}$

	$\sin(2\alpha) =$	$\cos(2\alpha) =$	$\tan(2\alpha) =$	$\cot(2\alpha) =$
e 61	$2\sin\alpha\cos\alpha$	$\cos^2\alpha-\sin^2\alpha$	$\frac{2\tan\alpha}{1-\tan^2\alpha}$	$\frac{\cot^2\alpha-1}{2\cot\alpha}$
e 62		$2\cos^2\alpha-1$	$\frac{2}{\cot\alpha-\tan\alpha}$	$\frac{1}{2}\cot\alpha-\frac{1}{2}\tan\alpha$
e 63		$1-2\sin^2\alpha$		

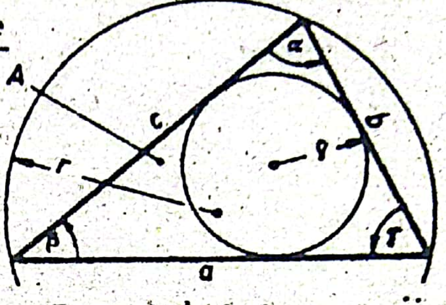
	$\sin\frac{\alpha}{2} =$	$\cos\frac{\alpha}{2} =$	$\tan\frac{\alpha}{2} =$	$\cot\frac{\alpha}{2} =$
e 64			$\frac{\sin\alpha}{1+\cos\alpha}$	$\frac{\sin\alpha}{1-\cos\alpha}$
e 65	$\sqrt{\frac{1-\cos\alpha}{2}}$	$\sqrt{\frac{1+\cos\alpha}{2}}$	$\frac{1-\cos\alpha}{\sin\alpha}$	$\frac{1+\cos\alpha}{\sin\alpha}$
e 66			$\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$	$\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}$

FUNCTIONS OF A CIRCLE

Acute angle triangle

E 6

Oblique angle triangle



Sine Rule

e 67 $\sin\alpha : \sin\beta : \sin\gamma = a : b : c$

e 68 $a = \dfrac{b}{\sin\beta}\sin\alpha = \dfrac{c}{\sin\gamma}\sin\alpha$

e 69 $b = \dfrac{a}{\sin\alpha}\sin\beta = \dfrac{c}{\sin\gamma}\sin\beta$

e 70 $c = \dfrac{a}{\sin\alpha}\sin\gamma = \dfrac{b}{\sin\beta}\sin\gamma$

Cosine Rule

e 71 $a^2 = b^2 + c^2 - 2\,bc\cos\alpha$

e 72 $b^2 = c^2 + a^2 - 2\,ac\cos\beta$

e 73 $c^2 = a^2 + b^2 - 2\,ab\cos\gamma$

(for obtuse angles the cosine is negative)

Tangent Rule

e 74 $\dfrac{a+b}{a-b} = \dfrac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}} \quad \Big| \quad \dfrac{a+c}{a-c} = \dfrac{\tan\frac{\alpha+\gamma}{2}}{\tan\frac{\alpha-\gamma}{2}} \quad \Big| \quad \dfrac{b+c}{b-c} = \dfrac{\tan\frac{\beta+\gamma}{2}}{\tan\frac{\beta-\gamma}{2}}$

Half-angle Rule

e 75 $\tan\dfrac{\alpha}{2} = \dfrac{\rho}{s-a} \quad \Big| \quad \tan\dfrac{\beta}{2} = \dfrac{\rho}{s-b} \quad \Big| \quad \tan\dfrac{\gamma}{2} = \dfrac{\rho}{s-c}$

Area, radius of incircle and circumcircle

e 76 $A = \dfrac{1}{2}\,bc\sin\alpha = \dfrac{1}{2}\,ac\sin\beta = \dfrac{1}{2}\,ab\sin\gamma$

77 $A = \sqrt{s(s-a)(s-b)(s-c)} = \rho\, s$

78 $\rho = \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$

e 79 $r = \dfrac{1}{2}\,\dfrac{a}{\sin\alpha} = \dfrac{1}{2}\,\dfrac{b}{\sin\beta} = \dfrac{1}{2}\,\dfrac{c}{\sin\gamma}$

e 80 $s = \dfrac{a+b+c}{2}$

FUNCTIONS OF A CIRCLE

Inverse trigonometric functions

E7

Inverse circular functions

Definitions

	function $y =$			
	$\arcsin x$	$\arccos x$	$\arctan x$	$\operatorname{arccot} x$
identical with	$x = \sin y$	$x = \cos y$	$x = \tan y$	$x = \cot y$
defined within	$-1 \leqq x \leqq +1$	$-1 \leqq x \leqq +1$	$-\infty < x < +\infty$	$-\infty < x < +\infty$
principal value	$-\frac{\pi}{2} \leqq y \leqq +\frac{\pi}{2}$	$\pi \geqq y \geqq 0$	$-\frac{\pi}{2} < y < +\frac{\pi}{2}$	$\pi > y > 0$

Basic properties

$$\arccos x = \frac{\pi}{2} - \arcsin x \qquad \Big| \qquad \operatorname{arccot} x = \frac{\pi}{2} - \arctan x$$

Ratios between inverse circular functions

For $x \geqq 0$:

$\arcsin x =$	$\arccos x =$	$\arctan x =$	$\operatorname{arccot} x =$
$\arccos\sqrt{1-x^2}$	$\arcsin\sqrt{1-x^2}$	$\arcsin\frac{x}{\sqrt{1+x^2}}$	$\arcsin\frac{1}{\sqrt{1+x^2}}$
$\arctan\frac{x}{\sqrt{1-x^2}}$	$\arctan\frac{\sqrt{1-x^2}}{x}$	$\arccos\frac{1}{\sqrt{1+x^2}}$	$\arccos\frac{x}{\sqrt{1+x^2}}$
$\operatorname{arccot}\frac{\sqrt{1-x^2}}{x}$	$\operatorname{arccot}\frac{x}{\sqrt{1-x^2}}$	$\operatorname{arccot}\frac{1}{x}$	$\arctan\frac{1}{x}$

For $x \leqq 0$:

$$\arcsin(-x) = -\arcsin x \qquad \Big| \qquad \arccos(-x) = \pi - \arccos x$$

$$\arctan(-x) = -\arctan x \qquad \Big| \qquad \operatorname{arccot}(-x) = \pi - \operatorname{arccot} x$$

Addition theorems

$$\arcsin a \pm \arcsin b = \arcsin\left(a\sqrt{1-b^2} \pm b\sqrt{1-a^2}\right)$$

$$\arccos a \pm \arccos b = \arccos\left(a\,b \mp \sqrt{1-a^2}\,\sqrt{1-b^2}\right)$$

$$\arctan a \pm \arctan b = \arctan\frac{a \pm b}{1 \mp a\,b}$$

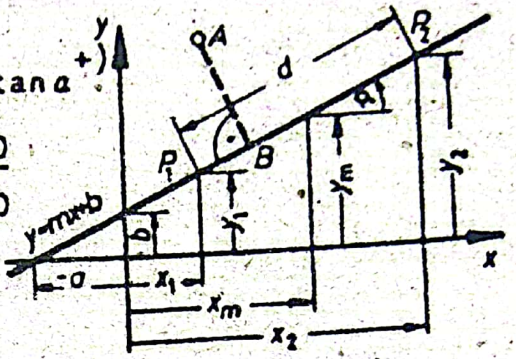
$$\operatorname{arccot} a \pm \operatorname{arccot} b = \operatorname{arccot}\frac{a\,b \mp 1}{b \pm a}$$

ANALYTICAL GEOMETRY

F1

Straight line, Triangle

Straight line



f 1 **Equation** $y = mx + b$

f 2 **Gradient** $m = \frac{y_2 - y_1}{x_2 - x_1} = \tan\alpha^{+)}$

Interc. form for $a \neq 0;\ b \neq 0$

f 3 $$\frac{x}{a} + \frac{y}{b} - 1 = 0$$

Gradient m_l of perpendicular AB

f 4 $$m_l = \frac{-1}{m}$$

Line joining two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$

f 5 $$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

Line through one point $P_1(x_1, y_1)$ and gradient m

f 6 $$y - y_1 = m(x - x_1)$$

f 7 **Distance between two points** $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}^{\,+)}$

Mid point of a line joining two points

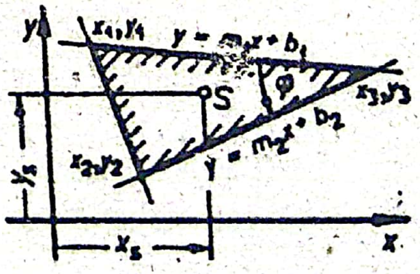
f 8 $$x_m = \frac{x_1 + x_2}{2} \quad \bigg| \quad y_m = \frac{y_1 + y_2}{2}$$

Point of intersection of two straight lines (see diagram triangle)

f 9 $$x_3 = \frac{b_2 - b_1}{m_1 - m_2} \quad \bigg| \quad y_3 = m_1 x_3 + b_1 = m_2 x_3 + b_2$$

f10 **Angle of intersection φ of two straight lines** $\tan\varphi = \frac{m_2 - m_1}{1 + m_2 \cdot m_1}^{+)}$ (see diagram-triangle)

Triangle



f11 **Centroid S** $x_s = \frac{x_1 + x_2 + x_3}{3}$

f12 $y_s = \frac{y_1 + y_2 + y_3}{3}$

Area

$$A = \frac{(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3)}{2}$$

f13 +) Where x and y have same dimension and are represented in equal scales (see also h 1).

ANALYTICAL GEOMETRY

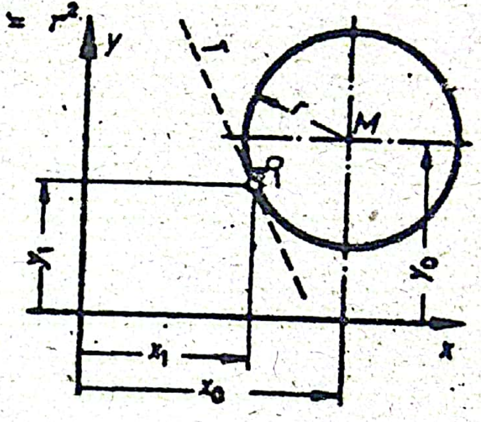
Circle, Parabola

F 2

Circle

Circle ecuation

	centre	
	at the origin	elsewhere
f 14	$x^2 + y^2 = r^2$	$(x-x_0)^2 + (y-y_0)^2 = r^2$



Basic equation

f 15 $$x^2 + y^2 + ax + by + c = 0$$

Radius of circle

f 16 $$r = \sqrt{x_0^2 + y_0^2 - c}$$

Coordinates of the centre M

f 17 $$x_0 = -\frac{a}{2} \quad \Big| \quad y_0 = -\frac{b}{2}$$

Tangent T at point $P_1(x_1, y_1)$

f 18 $$y = \frac{r^2 - (x-x_0)(x_1-x_0)}{y_1 - y_0} + y_0$$

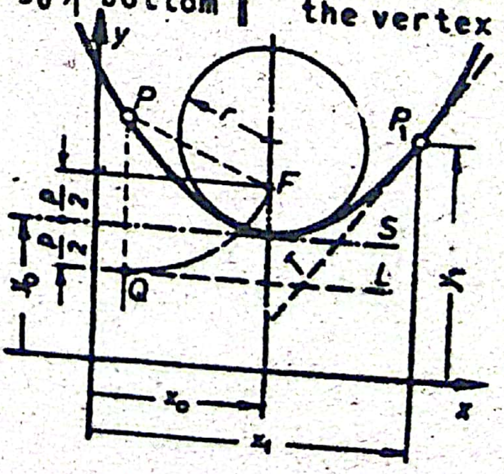
Parabola

Parabola equation

(by converting to this equation the vertex and parameter p may be ascertained)

	vertex		parabola open at
	at the origin	elsewhere	
f 19	$x^2 = 2py$	$(x-x_0)^2 = 2p(y-y_0)$	top
f 20	$x^2 = -2py$	$(x-x_0)^2 = -2p(y-y_0)$	bottom

F: focus
L: directrix
S: tangent at the vertex



Basic equation

f 21 $$y = ax^2 + bx + c$$

f 22 **Vertex radius** $r = p$

f 23 **Basic property** $\overline{PF} = \overline{PQ}$

Tangent T at point $P_1(x_1, y_1)$

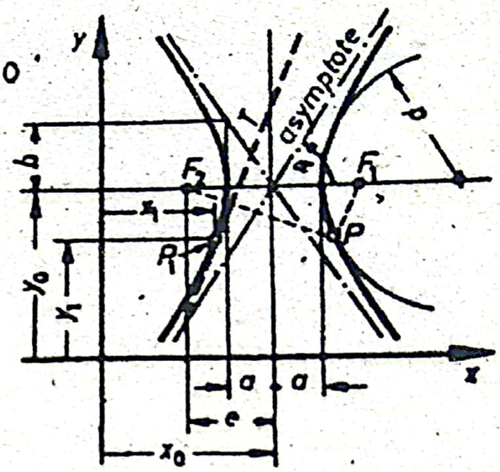
f 24 $$y = \frac{2(y_1 - y_0)(x - x_1)}{x_1 - x_0} + y_1$$

Hyperbola

Hyperbolic equation

point of intersection of asymptotes

	at the origin	elsewhere
f 25	$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$	$\frac{(x - x_0)^2}{a^2} - \frac{(y - y_0)^2}{b^2} - 1 = 0$



Basic equation

f 26 $ax^2 + by^2 + cx + dy + e = 0$

Basic property

f 27 $\overline{F_2P} - \overline{F_1P} = 2a$

Eccentricity

f 28 $e = \sqrt{a^2 + b^2}$ +)

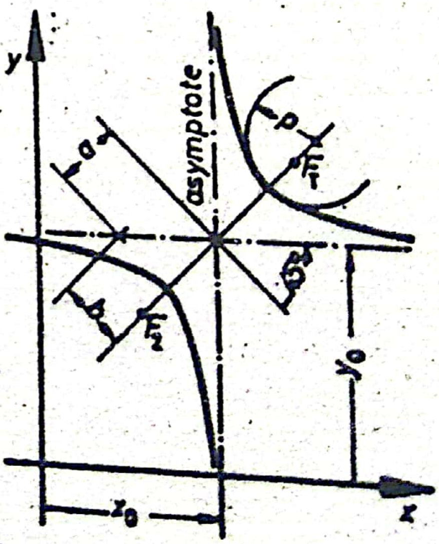
Gradient of asymptotes

f 29 $\tan\alpha = m = \pm\frac{b}{a}$ +)

Vertex radius $p = \frac{b^2}{a}$

f 30 **Tangent T** at $P_1(x_1, y_1)$ $y = \frac{b^2}{a^2}\,\frac{(x_1 - x_0)(x - x_1)}{y_1 - y_0} + y_1$

Rectangular hyperbola



Explanation in a rectangular hyperbola $a = b$ ..thus

Gradient of asymptotes

f 31 $\tan\alpha^{+)} = m = \pm 1$ $(\alpha = 45^\circ)$

Equation (for asymptotes parallel to x and y axes):
point of intersection
of asymptotes

	at the origin	elsewhere
f 32	$x\,y = c^2$	$(x-x_0)(y-y_0) = c^2$

Vertex radius

f 33 $p = a$ (parameter)

+) Conditions according to note on page F 1

ANALYTICAL GEOMETRY

Ellipse, Exponential curve

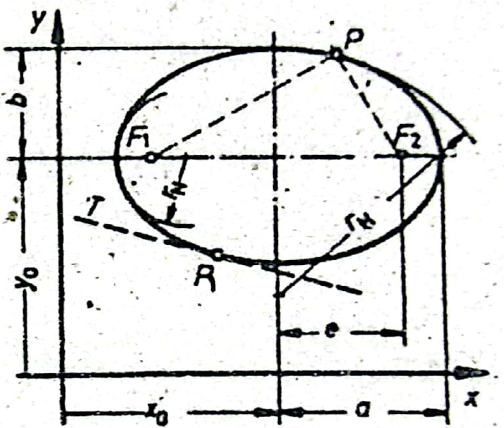
Ellipse

Ellipse equation

	point of intersection of axes	
	at the origin	elsewhere
f 34	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$	$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}-1\neq 0$

Vertex radii

f 35 $r_N = \frac{b^2}{a}$ | $r_H = \frac{a^2}{b}$



Eccentricity

f 36 $e = \sqrt{a^2-b^2}$

Basic property

f 37 $\overline{F_1P}+\overline{F_2P} = 2a$

Tangent T at $P_1\ (x_1;\ y_1)$

f 38 $y = -\frac{b^2}{a^2}\cdot\frac{(x_1-x_0)(x-x_1)}{y_1-y_0}+y_1$

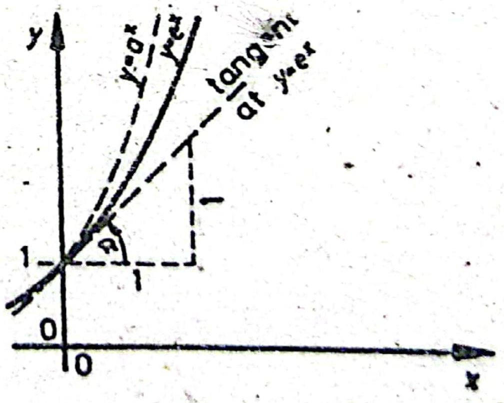
Note: F_1 and F_2 are focal points

Exponential curve

Basic equation

f 39 $y = a^x$

Here a is a positive constant, and x is a number.



Note

All exponential curves pass through the point $x = 0$; $y = 1$.

The derivative of the curve passing through this point with a gradient of 45° ($\tan\alpha^{+)} = 1$) is equal to the curve itself. The constant a now becomes e (Euler number) and is the base of the natural log.

$$e = 2{\cdot}718\,281\,828\,459\ldots$$

+) Conditions according to note on page F 1

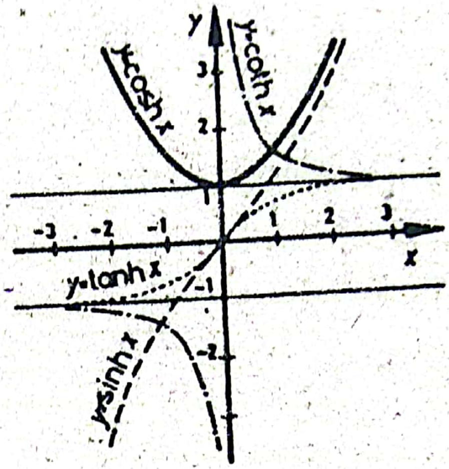
HYPERBOLIC FUNCTIONS

Hyperbolic functions

G1

Hyperbolic functions

Definition



g 1 $\sinh x = \frac{e^x - e^{-x}}{2}$

g 2 $\cosh x = \frac{e^x + e^{-x}}{2}$

g 3 $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$

g 4 $\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x} + 1}{e^{2x} - 1}$

Basic properties

g 5 $\cosh^2 x - \sinh^2 x = 1$

g 6 $\tanh x \times \coth x = 1$

g 7 $\tanh x = \frac{\sinh x}{\cosh x} \;\Big|\; 1 - \tanh^2 x = \frac{1}{\cosh^2 x} \;\Big|\; 1 - \coth^2 x = \frac{-1}{\sinh^2 x}$

Ratios between hyperbolic functions

where x is positive

	$\sinh x =$	$\cosh x =$	$\tanh x =$	$\coth x =$
g 8	$\sqrt{\cosh^2 x - 1}$	$\sqrt{\sinh^2 x + 1}$	$\frac{\sinh x}{\sqrt{\sinh^2 x + 1}}$	$\frac{\sqrt{\sinh^2 x + 1}}{\sinh x}$
g 9	$\frac{\tanh x}{\sqrt{1 - \tanh^2 x}}$	$\frac{1}{\sqrt{1 - \tanh^2 x}}$	$\frac{\sqrt{\cosh^2 x - 1}}{\cosh x}$	$\frac{\cosh x}{\sqrt{\cosh^2 x - 1}}$
g 10	$\frac{1}{\sqrt{\coth^2 x - 1}}$	$\frac{\coth x}{\sqrt{\coth^2 x - 1}}$	$\frac{1}{\coth x}$	$\frac{1}{\tanh x}$

For negative argument

g 11 $\sinh(-x) = -\sinh x \;\Big|\; \cosh(-x) = +\cosh x$

g 12 $\tanh(-x) = -\tanh x \;\Big|\; \coth(-x) = -\coth x$

Addition theorems

g 13 $\sinh(a \pm b) = \sinh a \cosh b \pm \cosh a \sinh b$

g 14 $\cosh(a \pm b) = \cosh a \cosh b \pm \sinh a \sinh b$

g 15 $\tanh(a \pm b) = \frac{\tanh a \pm \tanh b}{1 \pm \tanh a \tanh b}$

g 16 $\coth(a \pm b) = \frac{\coth a \coth b \pm 1}{\coth a \pm \coth b}$

Inverse hyperbolic functions

Definition

		function $y =$			
		$\operatorname{arsinh} x$	$\operatorname{arcosh} x$	$\operatorname{artanh} x$	$\operatorname{arcoth} x$
g 17	identical with	$x = \sinh y$	$x = \cosh y$	$x = \tanh y$	$x = \coth y$
g 18	logarithmic equivalents	$= \ln(x+\sqrt{x^2+1})$	$= \ln(x+\sqrt{x^2-1})$	$= \frac{1}{2}\ln\frac{1+x}{1-x}$	$= \frac{1}{2}\ln\frac{x+1}{x-1}$
g 19	defined within	$-\infty < x < +\infty$	$1 \leq x < +\infty$	$\lvert x\rvert < 1$	$\lvert x\rvert > 1$
g 20	primary value	$-\infty < y < +\infty$	$0 \leq y < +\infty$	$-\infty < y < +\infty$	$-\infty < y < +\infty$

Ratios between inverse hyperbolic functions

where x is positive:

	$\operatorname{arsinh} x =$	$\operatorname{arcosh} x =$	$\operatorname{artanh} x =$	$\operatorname{arcoth} x =$
g 21	$\operatorname{arcosh}\sqrt{1+x^2}$	$\operatorname{arsinh}\sqrt{x^2-1}$	$\operatorname{arsinh}\frac{x}{\sqrt{1-x^2}}$	$\operatorname{arsinh}\frac{1}{\sqrt{x^2-1}}$
g 22	$\operatorname{artanh}\frac{x}{\sqrt{1+x^2}}$	$\operatorname{artanh}\frac{\sqrt{x^2-1}}{x}$	$\operatorname{arcosh}\frac{1}{\sqrt{1-x^2}}$	$\operatorname{arcosh}\frac{x}{\sqrt{x^2-1}}$
g 23	$\operatorname{arcoth}\frac{\sqrt{1+x^2}}{x}$	$\operatorname{arcoth}\frac{x}{\sqrt{x^2-1}}$	$\operatorname{arcoth}\frac{1}{x}$	$\operatorname{artanh}\frac{1}{x}$

For negative argument:

g 24 $\operatorname{arsinh}(-x) = -\operatorname{arsinh} x$

g 25 $\operatorname{artanh}(-x) = -\operatorname{artanh} x$ $\qquad$ $\operatorname{arcoth}(-x) = -\operatorname{arcoth} x$

Addition theorems

g 26 $\operatorname{arsinh} a \pm \operatorname{arsinh} b = \operatorname{arsinh}\left(a\sqrt{b^2+1} \pm b\sqrt{a^2+1}\right)$

g 27 $\operatorname{arcosh} a \pm \operatorname{arcosh} b = \operatorname{arcosh}\left[ab \pm \sqrt{(a^2-1)(b^2-1)}\right]$

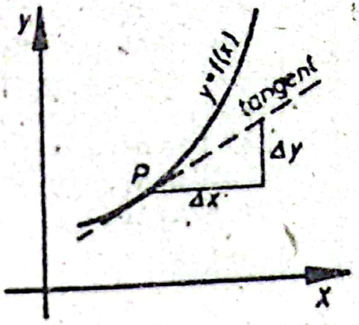
g 28 $\operatorname{artanh} a \pm \operatorname{artanh} b = \operatorname{artanh}\frac{a \pm b}{1 \pm ab}$

g 29 $\operatorname{arcoth} a \pm \operatorname{arcoth} b = \operatorname{arcoth}\frac{ab \pm 1}{a \pm b}$

Differential coefficients (or derivatives)

Gradient of a curve

The gradient of a curve $y = f(x)$ varies from point to point. By the gradient of a curve at point P we mean the gradient of the tangent at the point. If x and y have equal dimensions - which is not the case in most technical diagrams - and are presented at equal scales, the gradient may be expressed by the tangent of angle α between the tangent at point P and the horizontal axis:



$$m = \tan\alpha$$

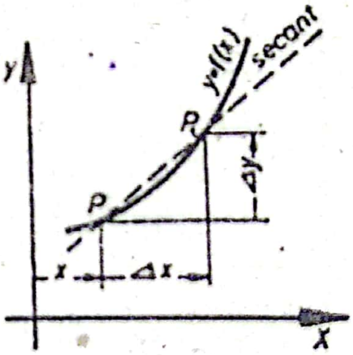
Always applicable is:

h 1 $$\text{gradient } m = \frac{\Delta y}{\Delta x}.$$

Difference coefficient

The difference coefficient or mean gradient of the function $y = f(x)$ between PP_1 is:

h 2 $$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$



Differential coefficient

Where Δx is infinitely small, i.e. where Δx approches zero, the slope at P becomes the limiting value of the slope of one of the secants. This slope is the "derivative" or "differential coefficient" of the function at P.

h 3 $$y' = \frac{dy}{dx} = f'(x)$$

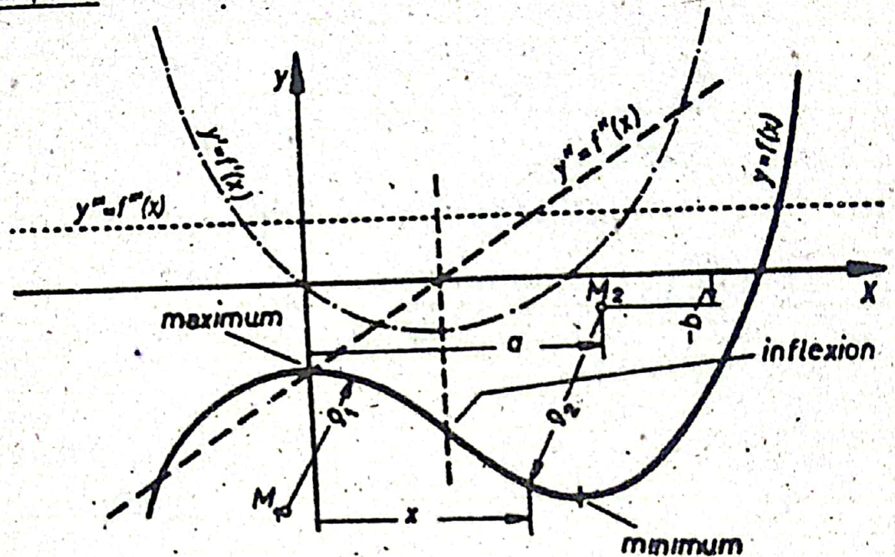
$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{dy}{dx} = f'(x)$$

Geometric meaning of derivative

Gradient of a curve

If, for each point x of a curve, we plot its corresponding gradient as an ordinate y', we obtain the first gradient curve $y' = f'(x)$ or the first derivative of the original curve $y = f(x)$. If we now take the derivative of the first gradient $y' = f'(x)$, we obtain $y'' = f''(x)$ or the second derivative of the original curve $y = f(x)$ etc.

Example: $y = Ax^3 + Bx^2 + Cx + D$



Radius of curvature ρ at any point x

h 4 $$\rho = \frac{\sqrt{(1 + y'^2)^3}}{y''}$$

M is below the curve where ρ is −
M is above the curve where ρ is +

Centre coordinates for radius ρ

h 5 $$a = x - \frac{1 + y'^2}{y''} y'$$

h 6 $$b = y + \frac{1 + y'^2}{y''}$$

continued on H 3

Determination of minima, maxima and inflexions

Minima and maxima

The value $x = a$ obtained for $y' = 0$ is inserted in y''.

h 7 For $y''(a) > 0$ there is a minimum at $x = a$,
h 8 for $y''(a) < 0$ there is a maximum at $x = a$,
h 9 for $y''(a) = 0$ see h 19.

Inflexion

The value $x = a$ obtained for $y'' = 0$ is inserted in y'''.

h 10 For $y'''(a) \neq 0$ there is an inflexion at $x = a$.

Shape of the curve $y = f(x)$

Rise and fall

h 11	$y'(x) > 0$	$y(x)$ increases as x increases
h 12	$y'(x) < 0$	$y(x)$ decreases as x increases
h 13	$y'(x) = 0$	$y(x)$ is tangentially parallel the x-axis at x

Curve

h 14	$y''(x) < 0$	$y(x)$ is convex (viewed from above)
h 15	$y''(x) > 0$	$y(x)$ is concave (viewed from above)

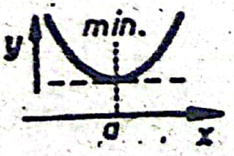
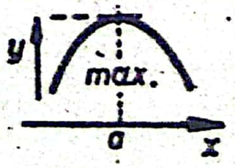
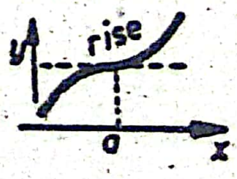
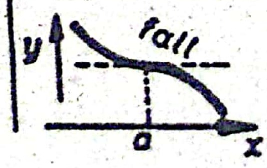
h 16	$y''(x) = 0$	with	a change of sign	flexion
		without	$y'(x)$ at x has a	bottom point

Exceptional case

Where at a point $x = a$

h 17 $y'(a) = y''(a) = y'''(a) = \ldots\ y^{(n-1)}(a) = 0$, but
h 18 $y^{n}(a) \neq 0$, one of the 4 conditions is present:

h 19

n = even number		n = uneven number	
$y^{(n)}(a) > 0$	$y^{(n)}(a) < 0$	$y^{(n)}(a) > 0$	$y^{(n)}(a) < 0$
			

Derivatives

Basic rules

	function	derivative
h 21	$y = c\,x^n + C$	$y' = c\,n\,x^{n-1}$
h 22	$y = u(x) \pm v(x)$	$y' = u'(x) \pm v'(x)$
h 23	$y = u(x)\,v(x)$	$y' = u'v + u\,v'$
h 24	$y = \dfrac{u(x)}{v(x)}$	$y' = \dfrac{u'v - u\,v'}{v^2}$
h 25	$y = \sqrt{x}$	$y' = \dfrac{1}{2\sqrt{x}}$
h 26	$y = u(x)^{v(x)}$	$y' = u^v\left(\dfrac{u'v}{u} + v'\ln u\right)$

Derivative of a function of a function (chain rule)

h 27 $y = f[u(x)]$ | $y' = f'(u)\;u'(x) = \dfrac{dy}{dx} = \dfrac{dy}{du}\,\dfrac{du}{dx}$

Parametric form of derivative

h 28 $y = f(x)$ $\quad \begin{cases} x = f(t) \\ y = f(t) \end{cases}$ | $y' = \dfrac{dy}{dt}\,\dfrac{dt}{dx} = \dfrac{\dot y}{\dot x}$

h 29 | $y'' = \dfrac{d^2y}{dx^2} = \dfrac{\dot x\ddot y - \dot y\ddot x}{\dot x^3}$

Derivative of inverse functions

The equation $y = f(x)$ solved for x gives the inverse function $x = \varphi(y)$.

h 30 $x = \varphi(y)$ | $f'(x) = \dfrac{1}{\varphi'(x)}$

Example

h 31 $y = f(x) = \arccos x$ | $f'(x) = \dfrac{1}{-\sin y} = -\dfrac{1}{\sqrt{1-x^2}}$

h 32 gives $x = \varphi(y) = \cos y$

DIFFERENTIAL CALCULUS

Basic differentials

H 5

Derivatives

Exponential functions

	function	derivative
h 33	$y = e^{x}$	$y' = e^{x} = y'' = \ldots$
h 34	$y = e^{-x}$	$y' = -e^{-x}$
h 35	$y = e^{ax}$	$y' = a\,e^{ax}$
h 36	$y = x\,e^{x}$	$y' = e^{x}\,(1 + x)$
h 37	$y = \sqrt{e^{x}}$	$y' = \frac{\sqrt{e^{x}}}{2}$
h 38	$y = a^{x}$	$y' = a^{x} \ln a$
h 39	$y = a^{nx}$	$y' = n\,a^{nx} \ln a$
h 40	$y = a^{x^2}$	$y' = a^{x^2}\,2x \ln a$

Trigonometrical functions

h 41	$y = \sin x$	$y' = \cos x$
h 42	$y = \cos x$	$y' = -\sin x$
h 43	$y = \tan x$	$y' = \frac{1}{\cos^2 x} = 1 + \tan^2 x$ $= \sec^2 x$
h 44	$y = \cot x$	$y' = \frac{-1}{\sin^2 x} = -(1 + \cot^2 x)$ $= -\mathrm{cosec}^2 x$
h 45	$y = a \sin(kx)$	$y' = a\,k \cos(kx)$
h 46	$y = a \cos(kx)$	$y' = -a\,k \sin(kx)$
h 47	$y = \sin^n x$	$y' = n \sin^{n-1} x \cos x$
h 48	$y = \cos^n x$	$y' = -n \cos^{n-1} x \sin x$
h 49	$y = \tan^n x$	$y' = n \tan^{n-1} x\,(1 + \tan^2 x)$
h 50	$y = \cot^n x$	$y' = -n \cot^{n-1} x\,(1 + \cot^2 x)$
h 51	$y = \frac{1}{\sin x}$	$y' = \frac{-\cos x}{\sin^2 x}$
h 52	$y = \frac{1}{\cos x}$	$y' = \frac{\sin x}{\cos^2 x}$

DIFFERENTIAL CALCULUS

H 6

Basic differentials

Derivatives

Logarithmic functions

	function	derivative
h 53	$y = \ln x$	$y' = \frac{1}{x}$
h 54	$y = {}^{a}\log x$	$y' = \frac{1}{x \ln a}$
h 55	$y = \ln(1 \pm x)$	$y' = \frac{\pm 1}{1 \pm x}$
h 56	$y = \ln x^{n}$	$y' = \frac{n}{x}$
h 57	$y = \ln \sqrt{x}$	$y' = \frac{1}{2x}$

Hyperbolic functions

h 58	$y = \sinh x$	$y' = \cosh x$
h 59	$y = \cosh x$	$y' = \sinh x$
h 60	$y = \tanh x$	$y' = \frac{1}{\cosh^{2} x}$
h 61	$y = \coth x$	$y' = \frac{-1}{\sinh^{2} x}$

Inverse trigonometrical functions

h 62	$y = \arcsin x$	$y' = \frac{1}{\sqrt{1 - x^{2}}}$
h 63	$y = \arccos x$	$y' = -\frac{1}{\sqrt{1 - x^{2}}}$
h 64	$y = \arctan x$	$y' = \frac{1}{1 + x^{2}}$
h 65	$y = \operatorname{arccot} x$	$y' = -\frac{1}{1 + x^{2}}$
h 66	$y = \operatorname{arsinh} x$	$y' = \frac{1}{\sqrt{x^{2} + 1}}$
h 67	$y = \operatorname{arcosh} x$	$y' = \frac{1}{\sqrt{x^{2} - 1}}$
h 68	$y = \operatorname{artanh} x$	$y' = \frac{1}{1 - x^{2}}$
h 69	$y = \operatorname{arcoth} x$	$y' = \frac{1}{1 - x^{2}}$

INTEGRAL CALCULUS

Integration

J 1

Integration

Integration reverse of differentiation

By integral calculus we mean the problem of finding a function $y = f(x)$, the derivative of $F(x)$ being equal to $f(x)$. Thus

j 1 $$F'(x) = \frac{dF(x)}{dx} = f(x)$$

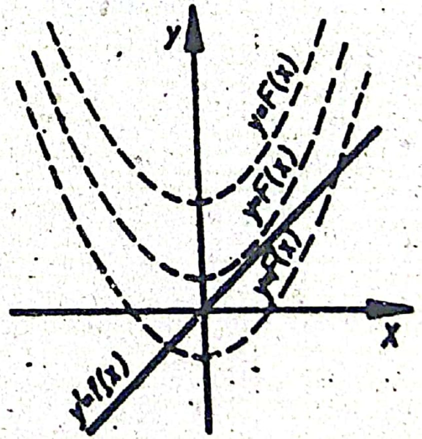
hence, by integration

the indefinite integral

j 2 $$\int f(x)\,dx = F(x) + C$$

Here C is unknown constant which disappears on differentiation, since the derivative of a constant equals zero.

Geometric meaning of the indefinite integral



As this figure shows, there are an infinite number of $y = F(x)$ curves with a gradient $y' = f(x)$. All $y = F(x)$ curves are the same shape, but intersect the y-axis at varying distances. The constant C, however, establishes a fixed curve. If the curve is to pass through the point x_0/y_0, then

j 3 $$C = y_0 - F(x_0)$$

The definite integral

The definite integral is represented by

j 4 $$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Here integration takes place between the limits a and b, the second substitution resultant is subtracted from the first causing the constant C to disappear.

Integration

Basic formulae

J 5 $$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad \text{here} \quad n \neq -1$$

J 6 $$\int \frac{dx}{x} = \ln x + C$$

J 7 $$\int \left[u(x) \pm v(x)\right] dx = \int u(x) \, dx \pm \int v(x) \, dx$$

J 8 $$\int \frac{u'(x)}{u(x)} \, dx = \ln u(x) + C$$

J 9 $$\int u(x) \, u'(x) \, dx = \frac{1}{2}\left[u(x)\right]^2 + C$$

Integration by parts

J10 $$\int u(x) \, v'(x) \, dx = u(x) \, v(x) - \int u'(x) \, v(x) \, dx$$

Integration by substitution

J11 $$\int f(x) \, dx = \int f\left[\varphi(z)\right] \varphi'(z) \, dz$$

here $x = \varphi(z)$ and $dx = \varphi'(z) \, dz$

Example:

J12 $F(x) = \int \sqrt{3x - 5} \, dx.$

Where $3x - 5 = z$, the derivative is $z = \frac{dz}{dx} = 3.$ Thus $dx = \frac{dz}{3}$ expressed in terms of z, the integral becomes $F(x) = \frac{1}{3}\int \sqrt{z} \, dz = \frac{2}{9} z\sqrt{z} + C.$ Insert value of z in above expression; $F(x) = \frac{2}{9}(3x-5)\sqrt{3x-5} + C$

INTEGRAL CALCULUS

Basic integrals

J 3

Integrals

(omitting integral constant C)

J 13 $\int e^x \, dx = e^x$ ‖ $\int \ln x \, dx = x \ln x - x$

J 14 $\int a^x \, dx = \frac{a^x}{\ln a}$ ‖ $\int \frac{dx}{x-a} = \ln(x-a)$

J 15 $\int \frac{dx}{(x-a)(x-b)} = \frac{1}{a-b} \ln \frac{x-a}{x-b} \qquad (a \neq b)$

J 16 $\int \frac{dx}{(x-a)^n} = -\frac{1}{(n-1)(x-a)^{n-1}} \qquad (n \neq 1)$

J 17 $\int \frac{dx}{x^2-a^2} = -\frac{1}{a} \operatorname{arcoth} \frac{x}{a} = \frac{1}{2a} \ln \frac{x-a}{x+a} \qquad (x > a)$

J 18 $\int \frac{dx}{a^2-x^2} = \frac{1}{a} \operatorname{artanh} \frac{x}{a} = \frac{1}{2a} \ln \frac{a+x}{a-x} \qquad (x < a)$

J 19 $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \arctan \frac{x}{a}$ ‖ $\int \frac{x \, dx}{x^2+a^2} = \frac{1}{2} \ln(x^2+a^2)$

J 20 $\int \frac{dx}{(x^2+a^2)^2} = \frac{x}{2a^2(x^2+a^2)} + \frac{1}{2a^3} \arctan \frac{x}{a}$

J 21 $\int \frac{dx}{(x^2+a^2)^n} = \frac{x}{2a^2(n-1)(x^2+a^2)^{n-1}} + \frac{2n-3}{2a^2(n-1)} \int \frac{dx}{(x^2+a^2)^{n-1}} \qquad (n \neq 1)$

J 22 $\int \sqrt{x} \, dx = \frac{2}{3} \sqrt{x^3}$ ‖ $\int \frac{dx}{\sqrt{x}} = 2\sqrt{x}$

J 23 $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a}$ ‖ $\int \frac{dx}{\sqrt{ax+b}} = \frac{2}{a} \sqrt{ax+b}$

J 24 $\int \frac{dx}{\sqrt{x^2-a^2}} = \operatorname{arcosh} \frac{x}{a} = \ln\left(x + \sqrt{x^2-a^2}\right)$

J 25 $\int \frac{dx}{\sqrt{x^2+a^2}} = \operatorname{arsinh} \frac{x}{a} = \ln\left(x + \sqrt{x^2+a^2}\right)$

J 26 $\int \sqrt{a^2-x^2} \, dx = \frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}$

J 27 $\int \sqrt{x^2-a^2} \, dx = \frac{x}{2} \sqrt{x^2-a^2} - \frac{a^2}{2} \operatorname{arcosh} \frac{x}{a}$

J 28 $\int \sqrt{x^2+a^2} \, dx = \frac{x}{2} \sqrt{x^2+a^2} + \frac{a^2}{2} \operatorname{arsinh} \frac{x}{a}$

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INTEGRAL CALCULUS

Basic integrals

Integrals

(omitting integral constant C)

j 29 $\int \sin x\,dx = -\cos x$

j 30 $\int \sin^2 x\,dx = \frac{x}{2} - \frac{1}{4}\sin(2x)$

j 31 $\int \sin^3 x\,dx = -\frac{3}{4}\cos x + \frac{1}{12}\cos(3x)$

j 32 $\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$

j 33 $\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax)$

---

j 34 $\int \cos x\,dx = \sin x$

j 35 $\int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{4}\sin(2x)$

j 36 $\int \cos^3 x\,dx = \frac{3}{4}\sin x + \frac{1}{12}\sin(3x)$

j 37 $\int \cos^n x\,dx = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$

j 38 $\int \cos(ax)\,dx = \frac{1}{a}\sin(ax)$

---

j 39 $\int \tan x\,dx = -\ln \cos x \quad\Big\|\quad \int \tan(ax)\,dx = -\frac{1}{a}\ln\cos(ax)$

j 40 $\int \tan^2 x\,dx = \tan x - x$

j 41 $\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx \qquad (n \neq 1)$

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j 42 $\int \cot x\,dx = \ln \sin x \quad\Big\|\quad \int \cot(ax)\,dx = \frac{1}{a}\ln\sin(ax)$

j 43 $\int \cot^2 x\,dx = -x - \cot x$

j 44 $\int \cot^n x\,dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\,dx \qquad (n \neq 1)$

---

j 45 $\int \frac{dx}{\sin x} = \ln \tan\frac{x}{2} \quad\Big\|\quad \int \frac{dx}{\sin^2 x} = -\cot x$

j 46 $\int \frac{dx}{\sin^n x} = -\frac{1}{n-1}\frac{\cos x}{\sin^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} x} \qquad (n \neq 1)$

j 47 $\int \frac{dx}{\cos x} = \ln \tan\left(\frac{x}{2} + \frac{x}{4}\right) \quad\Big\|\quad \int \frac{dx}{\cos^2 x} = \tan x$

j 48 $\int \frac{dx}{\cos^n x} = \frac{1}{n-1}\frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2} x} \qquad (n \neq 1)$

INTEGRAL CALCULUS

Basic integrals

J 5

Integrals

(omitting integral constant C)

j 49 $\int \frac{dx}{1 + \sin x} = \tan\left(\frac{x}{2} - \frac{\pi}{4}\right) \quad \Big| \quad \int \frac{dx}{1 - \sin x} = -\cot\left(\frac{x}{2} - \frac{\pi}{4}\right)$

j 50 $\int \frac{dx}{1 + \cos x} = \tan \frac{x}{2} \quad \Big| \quad \int \frac{dx}{1 - \cos x} = -\cot \frac{x}{2}$

j 51 $\int \sin(ax)\sin(bx) = -\frac{\sin(ax+bx)}{2(a+b)} + \frac{\sin(ax-bx)}{2(a-b)} \quad (|a| \neq |b|)$

j 52 $\int \sin(ax)\cos(bx) = -\frac{\cos(ax+bx)}{2(a+b)} - \frac{\cos(ax-bx)}{2(a-b)} \quad (|a| \neq |b|)$

j 53 $\int \cos(ax)\cos(bx) = \frac{\sin(ax+bx)}{2(a+b)} + \frac{\sin(ax-bx)}{2(a-b)} \quad (|a| \neq |b|)$

j 54 $\int x^n \sin(ax)\, dx = -\frac{x^n}{a}\cos(ax) + \frac{n}{a}\int x^{n-1} \cos(ax)\, dx$

j 55 $\int x^n \cos(ax)\, dx = \frac{x^n}{a}\sin(ax) - \frac{n}{a}\int x^{n-1} \sin(ax)\, dx$

---

j 56 $\int \arcsin x \, dx = x \arcsin x + \sqrt{1 - x^2}$

j 57 $\int \arccos x \, dx = x \arccos x - \sqrt{1 - x^2}$

j 58 $\int \arctan x \, dx = x \arctan x - \frac{1}{2} \ln(1 + x^2)$

j 59 $\int \operatorname{arccot} x \, dx = x \operatorname{arccot} x + \frac{1}{2} \ln(1 + x^2)$

---

j 60 $\int \sinh x \, dx = \cosh x$

j 61 $\int \sinh^2 x \, dx = \frac{1}{4}\sinh(2x) - \frac{x}{2}$

j 62 $\int \sinh^n x \, dx = \frac{1}{n}\cosh x \, \sinh^{n-1} x - \frac{n-1}{n}\int \sinh^{n-2} x \, dx$

j 63 $\int \sinh(ax)\, dx = \frac{1}{a}\cosh(ax)$

---

j 64 $\int \cosh x \, dx = \sinh x$

j 65 $\int \cosh^2 x \, dx = \frac{1}{4}\sinh(2x) + \frac{x}{2}$

j 66 $\int \cosh^n x \, dx = \frac{1}{n}\sinh x \, \cosh^{n-1} x + \frac{n-1}{n}\int \cosh^{n-2} x \, dx$

j 67 $\int \cosh(ax)\, dx = \frac{1}{a}\sinh(ax)$

Integrals

(omitting integral constant C)

J 63 $\int \tanh x \, dx = \ln \cosh x$

J 69 $\int \tanh^2 x \, dx = x - \tanh x$

J 70 $\int \tanh^n x \, dx = -\frac{1}{n-1} \tanh^{n-1} x + \int \tanh^{n-2} x \, dx \quad (n \neq 1)$

J 71 $\int \tanh(ax) \, dx = \frac{1}{a} \ln \cosh(ax)$

J 72 $\int \coth x \, dx = \ln \sinh x$

J 73 $\int \coth^2 x \, dx = x - \coth x$

J 74 $\int \coth^n x \, dx = -\frac{1}{n-1} \coth^{n-1} x + \int \coth^{n-2} x \, dx \quad (n \neq 1)$

J 75 $\int \coth(ax) \, dx = \frac{1}{a} \ln \sinh(ax)$

J 76 $\int \frac{dx}{\sinh x} = \ln \tanh \frac{x}{2}$

J 77 $\int \frac{dx}{\sinh^2 x} = -\coth x$

J 78 $\int \frac{dx}{\cosh x} = 2 \arctan e^x$

J 79 $\int \frac{dx}{\cosh^2 x} = \tanh x$

J 80 $\int \operatorname{arsinh} x \, dx = x \operatorname{arsinh} x - \sqrt{x^2 + 1}$

J 81 $\int \operatorname{arcosh} x \, dx = x \operatorname{arcosh} x - \sqrt{x^2 - 1}$

J 82 $\int \operatorname{artanh} x \, dx = x \operatorname{artanh} x + \frac{1}{2} \ln(1 - x^2)$

J 83 $\int \operatorname{arcoth} x \, dx = x \operatorname{arcoth} x + \frac{1}{2} \ln(x^2 - 1)$

J 84 $\int \sin^m x \cos^n x \, dx = \frac{1}{m+n} \sin^{m+1} x \cos^{n-1} x + \frac{n-1}{m+n} \int \sin^m x \cos^{n-2} x \, dx$

Where n is an uneven number the residual integral is:

J 85 $\int \sin^m x \cos x \, dx = \frac{\sin^{m+1} x}{m+1}$

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INTEGRAL CALCULUS

Application of integration

J 7

Arc differential $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

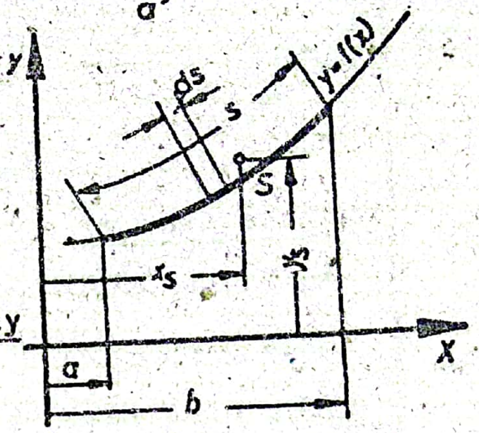
	arc length	surface area where the curve rotates around the x-axis
J 86	$s = \int_a^b \sqrt{1 + y'^2}\,dx$	$A_m = 2\pi \int_a^b y\sqrt{1 + y'^2}\,dx$

static moment of a curve

	x-axis	y-axis
J 87	$M_x = \int_a^b y\sqrt{1+y'^2}\,dx$	$M_y = \int_a^b x\sqrt{1+y'^2}\,dx$

coordinates of centre of gravity

J 88	$x_s = \frac{M_y}{s}$	$y_s = \frac{M_x}{s}$



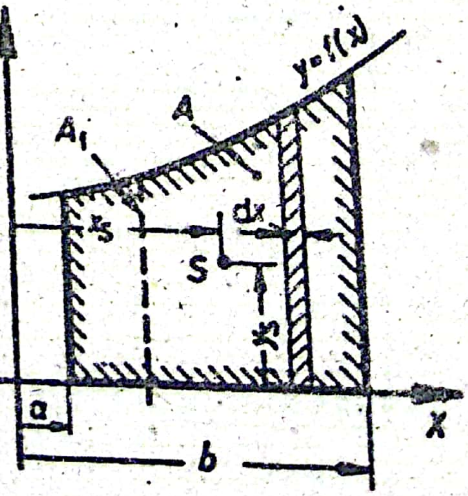
	area	rotating body where area A rotates around the x-axis	volume of a body, the cross section A_1 of which is a function of x
J 89	$A = \int_a^b y\,dx$	$V = \pi \int_a^b y^2\,dx$	$V = \int_a^b A_1(x)\,dx$

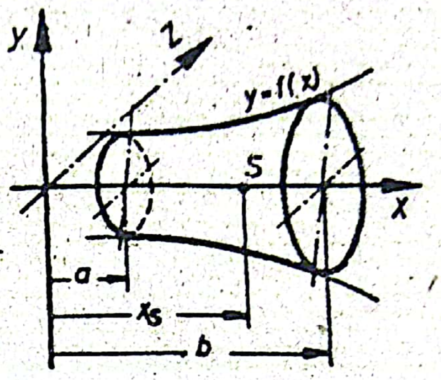
static moment of a curve in relation to the

	x-axis	y-axis
J 90	$H_x = \int_a^b \frac{y^2}{2}\,dx$	$H_y = \int_a^b xy\,dx$

coordinates of centre of gravity

J 91	$x_s = \frac{H_y}{A}$	$y_s = \frac{H_x}{A}$





Static moment of a body
(in relation to the y-z plane)

j 92 $$M_{yz} = \pi \int_a^b x \; y^2 \; dx$$

Coordinates of centre of gravity

j 93 $$x_s = \frac{M_{yz}}{V}$$

Pappus theorems

Surface area of a revolving body

A_m = arc length s times the distance covered by the centre of gravity

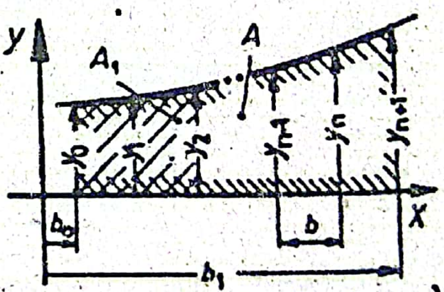
j 94 $= 2\pi s y_s$ (see also formulae j 86 and j 88)

Volume of a revolving body

V = area A times the distance covered by the centre of gravity

j 95 $= 2\pi A y_s$ (see also formulae j 89 and j 91)

Numerical integration



Division of area into an even number n of strips of equal width $h = \frac{b_1 - b_0}{n}$

j 96 Then, according to the

Trapezium rule

j 97 $$A = \frac{b}{2}(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + 2y_n + y_{n+1})$$

Simpson's rule for three ordinates:

j 98 $$A_1 = \frac{b}{3}(y_0 + 4y_1 + y_2)$$

Simpson's rule for more than three ordinates:

j 99 $$A = \frac{b}{3}\left[y_0 + y_n + 2(y_2 + y_4 + \ldots + y_{n-2}) + 4(y_1 + y_3 + \ldots + y_{n-1})\right]$$

INTEGRAL CALCULUS

Application of integration

J 9

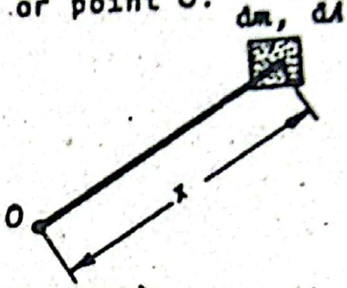
Moment of inertia

(Second moment of area)

General

By moment of inertia in relation to an axis x or a point O, we mean the sum of the products of line-, area-, volume- or mass-elements and the squares of their distances from the x-axis or point O:

	Moment of inertia	Second moment of area
J 100	$I = \int x^2 \, dm$ kg m^2	$I = \int x^2 \, dA$ m^4



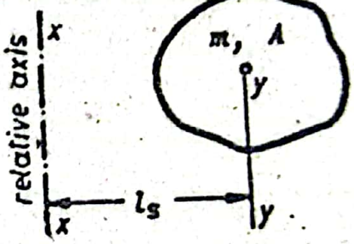
Steiner's theorem (Parallel axis theorem)

For every mass moment of inertia, both axial and polar, the following equation will apply:

J 101 $I_{xx} = I_{yy} + m \, l_s^2$ kg m^2

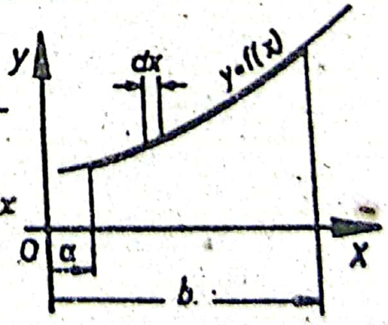
Similar equations will apply for line, area and volume moments of inertia:

$I_{xx} = I_{yy} + A \, l_s^2$ m^4



Moment of inertia of plane curves

	in relation to the x-axis	in relation to the y-axis
J 102	$I_{ox} = \int_a^b y^2 \sqrt{1+y'^2} \, dx$	$I_{oy} = \int_a^b x^2 \sqrt{1+y'^2} \, dx$



I_{xx} : moment of inertia about a general axis xx

I : moment of inertia about the centre of gravity

m, A : total length, area, volume, or mass

l_s : distance of centre of gravity from axis or point

INTEGRAL CALCULUS

Application of integration

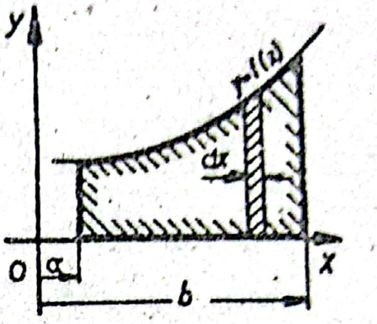
J 10

Moment of inertia of plane surface (see P 3)

Generally applicable

in relation to

	x-axis	y-axis
J 103	$I_{ox} = \int_a^b \frac{y^3}{3} dx$	$I_{oy} = \int_a^b x^2 y \, dx$



Polar second moment of area

Where the relative axes of the axial moments of inertia I_{ox} and I_{oy} are perpendicular to each other in any plane surface, the polar moment of inertia in relation to the pole or point of intersection O of the axis is:

J 104 $$J = I_{oz} = I_{ox} + I_{oy} \quad \text{m}^4$$

Second moments of area of a rectangle

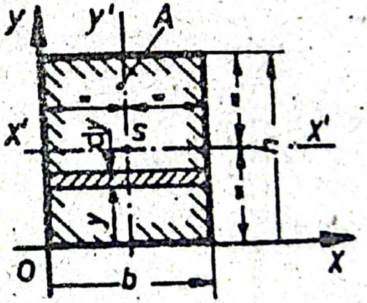
J 105 $$I_{ox} = \int_0^h y^2 b \, dy = b\left[\frac{y^3}{3}\right]_0^h = \frac{b\, h^3}{3}$$

J 106 $$I_{x'x'} = I_{ox} - A\left(\frac{h}{2}\right)^2 = \frac{b\, h^3}{12}$$

J 107 $$I_{oy} = \frac{b^3\, h}{3} \quad \Big\| \quad I_{y'y'} = \frac{b^3\, h}{12}$$

J 108 $$I_{oz} = I_{ox} + I_{oy} = \frac{b\, h^3}{3} + \frac{b^3\, h}{3} = \frac{b\, h}{3}(b^2 + h^2) \quad \text{m}^4$$

$$I_{z'z'} = I_{x'x'} + I_{y'y'} = \frac{b\, h^3}{12} + \frac{b^3\, h}{12} = \frac{b\, h}{12}(b^2 + h^2) \quad \text{m}^4$$

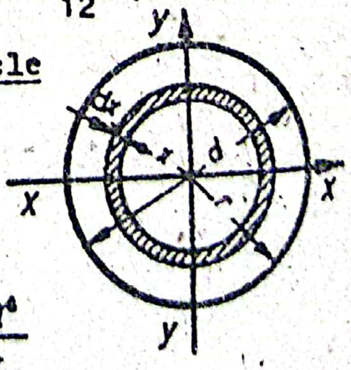


Polar 'moment of inertia' of a circle

J 109 $$J = \int_0^r x^2\, 2\pi x \, dx = 2\pi\left[\frac{x^4}{4}\right]_0^r = \frac{\pi\, r^4}{2}$$

Second moment of area of a circle about a diameter

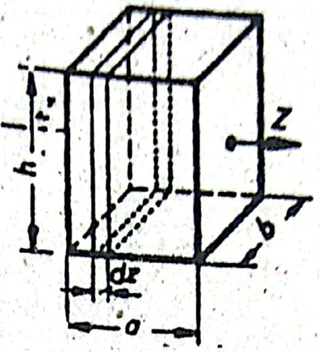
J 110 $$I_{xx} = I_{yy} = \frac{J}{2} = \frac{\pi\, r^4}{4} = \frac{\pi\, d^4}{64}$$



Second moment of volume of a solid

'Moment of inertia' of a cuboid

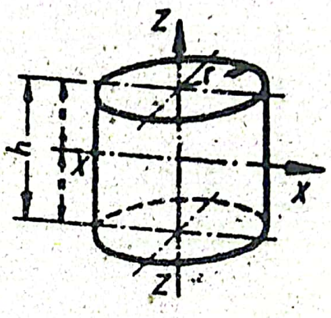
Where $\left(\frac{b\,h^3}{12}+\frac{b^3\,h}{12}\right)$ is the polar moment of inertia of a rectangle (see J 10), the equation for the z-axis:



J 111 $$I_{v,zz}=\int_0^a\left(\frac{bh^3}{12}+\frac{b^3h}{12}\right)dz=\frac{abh}{12}(b^2+h^2)$$

'Moment of inertia' of a circular cylinder

for the axis zz:



J 112 $$I_{v,zz}=\int_{-\frac{h}{2}}^{+\frac{h}{2}}\frac{\pi\ r^4}{4}\,dz=\frac{\pi\ r^4\ h}{2}$$

for the axis xx:

J 113 $$I_{v,xx}=\int_{-\frac{h}{2}}^{+\frac{h}{2}}\left(\frac{\pi\ r^4}{4}+\pi\ r^2 z^2\right)dz=\frac{\pi\ r^2 h}{12}(3r^2+h^2)$$

Dynamic moment of inertia (mass moment of inertia)

The mass moment of inertia I about a particular axis is the product of the second moment of volume I_v about that axis and the density ϱ.

J 114 $$I=I_v\times\varrho \qquad \text{kg m}^2,\ \text{kgf m s}^2,\quad \text{lb ft}^2$$

J 115 where $$\varrho=\frac{m}{V} \qquad \text{kg m}^{-3},\ \text{kg dm}^{-3},\quad \text{lb ft}^{-3}$$

e.g. for a cylinder about the axis ZZ:

J 116 $$I_{zz}=I_{v,zz}\times\frac{m}{V}=\frac{\pi\,r^4\,h}{2}\times\frac{m}{r^2\,\pi\,h}=\frac{m\,r^2}{2}$$

For other mass moments of inertia see

STATICS
General terms

General

Statics deals with the theory of equilibrium and with the determination of external forces acting on stationary solid bodies (e.g. support reactions). The contents of page K1...K14 are applicable only to forces acting in one plane.

The most important quantities of statics and their units

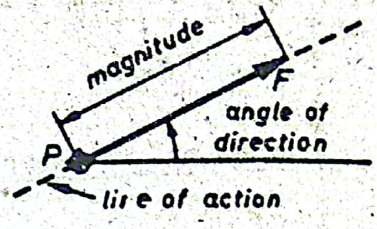
Length l

Is a base quantity, see preface.

Units: m; cm; km

Force F (see explanation on M 1)

Being a vector a force is defined by its magnitude, direction, and point of application (P).



Gravitational force G

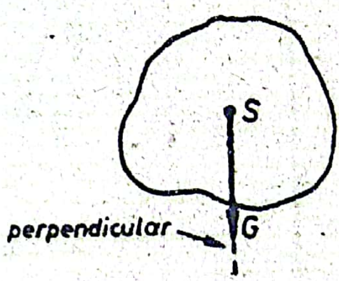
definition: force of earth's attract.

point of applicat: centre of grav. S

line of action: vertical line intersecting centre of gravity

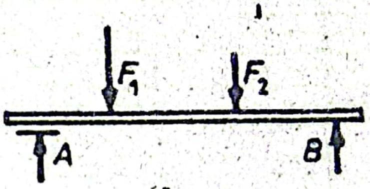
direction: downwards (towards earth's centre)

magnitude: determined by spring balance.



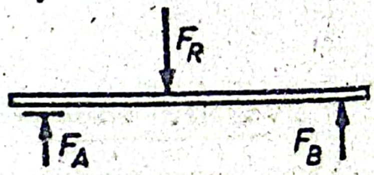
Support reaction F_A

force applied to body by support A.



Resultant force F_R

force representing the total action of several external forces.

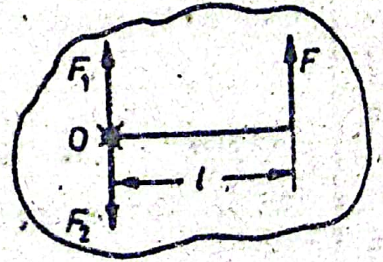


Moment M of a force F about a point O

The perpendicular distance from point O to the line of action of force F is called lever arm l.

F_1 and F_2 form a pair of forces.

The moment may be represented by a vector.



k 1 $$F_1 = F \; ; \quad F_2 = -F_1$$

k 2 moment $$M = F\,l$$

Moment theorem: The moment of the resultant force is equal to the sum of the moments of the individual forces.

STATICS

Forces composition

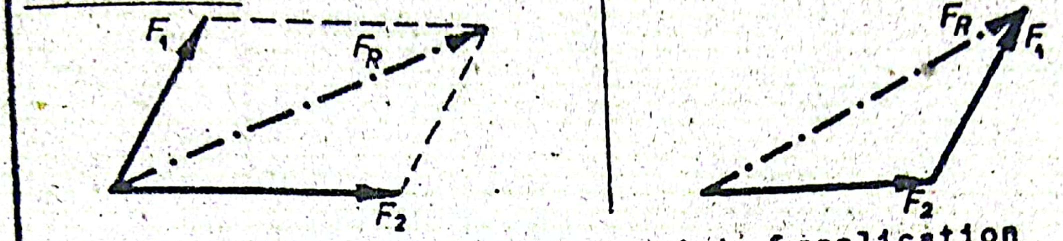
K 2

Graphical composition of forces

diagram of forces | force polygon

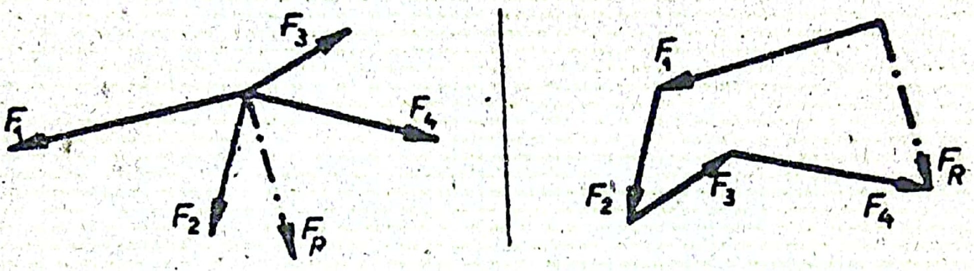
Two forces

k 4



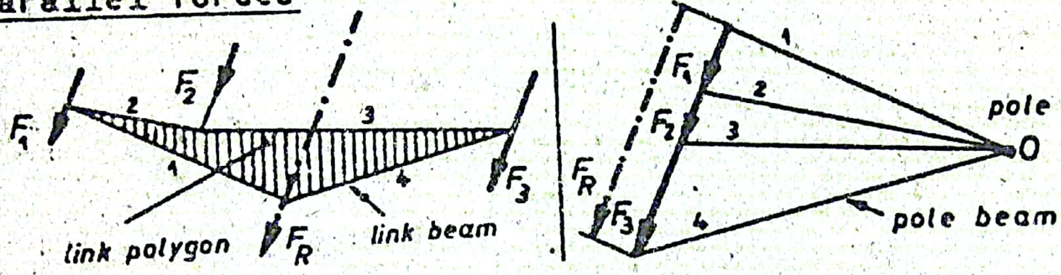
A number of forces and a common point of application

k 5



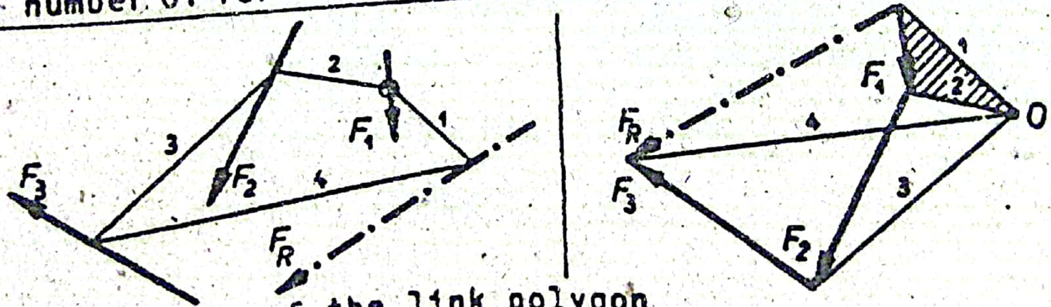
Parallel forces

k 6



A number of forces and any random point of application

k 7



Construction of the link polygon

Draw force polygon and determine pole O so as to avoid any link rays running parallel. Draw pole rays. Construct link polygon such that link rays run parallel to corresponding pole rays. Thereby each point of interection in the link polygon corresponds to a triangle in the force polygon (e.g. triangle F_1-1-2 of force polygon corresponds to point of intersection F_1-1-2 of link polygon).

STATICS

Forces composition

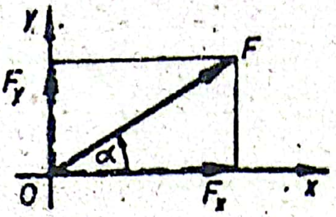
K 3

Mathematical composition of forces

Resolution of a force

k 8 $F_x = F\cos\alpha$ | $F_y = F\sin\alpha$

k 9 $F = +\sqrt{F_x^2 + F_y^2}$ | $\tan\alpha = \dfrac{F_y}{F_x}$,

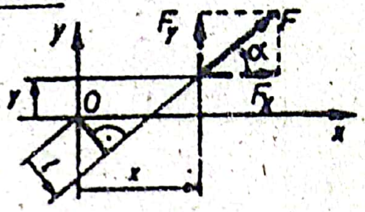


(for signs of trigonometrical functions of α see tables k 16 to k 19)

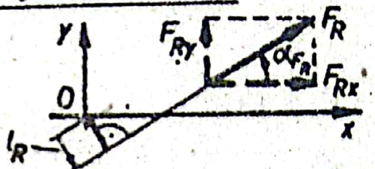
Moment M_O of a force about a point O

k 10 $M_O = \pm F\,l = F_y\,x - F_x\,y$

(for signs of trigonom. funct. of α see table k16 through k19)



Resultant force F_R of any random given forces



k 11 components $F_{Rx} = \Sigma F_x$ | $F_{Ry} = \Sigma F_y$

k 12 magnitude $F_R = +\sqrt{F_{Rx}^2 + F_{Ry}^2}$

k 13 angle of direction α_{F_R} $\tan\alpha_{F_R} = \dfrac{F_{Ry}}{F_{Rx}}$; $\sin\alpha_{F_R} = \dfrac{F_{Ry}}{F_R}$; $\cos\alpha_{F_R} = \dfrac{F_{Rx}}{F_R}$

k 14 distance $l_R = \dfrac{|\Sigma M_O|}{|F_R|}$ (moment theorem)

k 15 sign of $F_R\,l_R$ = sign of ΣM_O

Signs of trigonometr. functions of x, y; F_x, F_y; F_{Rx}, F_{Ry}

	quadrant	α, α_{F_R}	$\cos\alpha$	$\sin\alpha$	$\tan\alpha$	x, F_x, F_{Rx}	y, F_y, F_{Ry}
k 16	I	0... 90°	+	+	+	+	+
k 17	II	90...180°	−	+	−	−	+
k 18	III	180...270°	−	−	+	−	−
k 19	IV	270...360°	+	−	−	+	−

F_x, F_y : components of F parallel t x-axis and y-axis
F_{Rx}, F_{Ry} : components of F_R parallel to x-axis and y-axis
x, y : coordinates of F
α, α_{F_R} : angles of F and F_R
l, l_{F_R} : distances of F and F_R from reference point

STATICS

Equilibrium

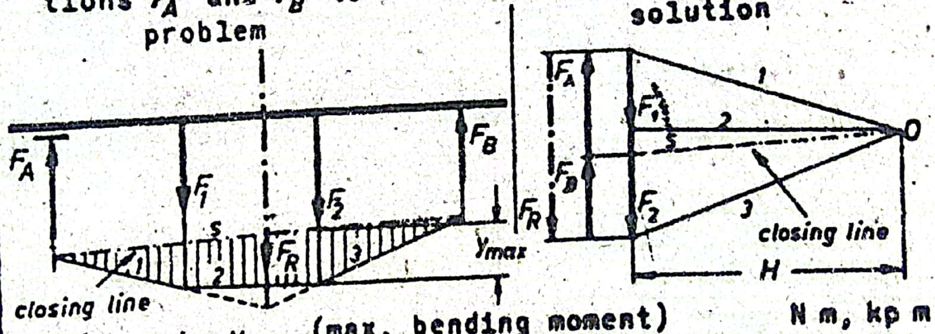
Conditions of equilibrium

A body is said to be in equilibrium when both the resultant force and the sum of the moments of all external forces about any random point are equal to zero.

	forces	graphical	mathematical
k 20	with common point of application	closed force polygon	$\Sigma F_x = 0; \Sigma F_y = 0$
k 21	parallel to vertical axis	force polygon and link polygon closed	$\Sigma F_y = 0; \Sigma M = 0$
k 22 k 23	arbitrary		$\Sigma F_x = 0; \Sigma F_y = 0$ $\Sigma M = 0$

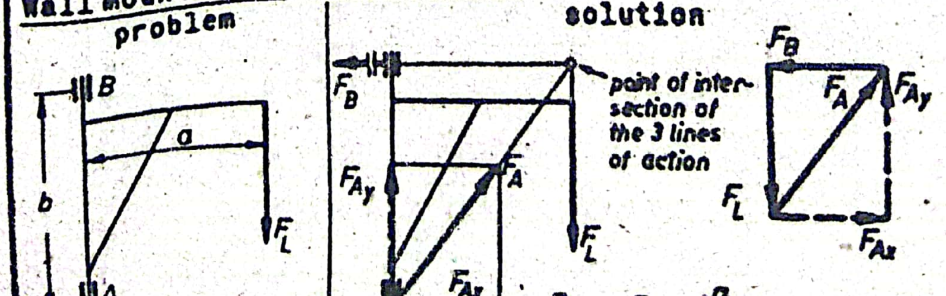
Examples

Girder on two supports with given load: support reactions F_A and F_B to be determined.



k 24
$M_{B\,max} = k_M \, y_{max}$ (max. bending moment) N m, kp m
$k_M = k_F \, k_L \, H$ (scale of moments) N m/m, kp m/mm
k_F : scale of forces N/m, kp/mm
k_L : scale of lengths m/m, m/mm
H : pole spacing m, cm, mm
F_R : resultant of F_1 and F_2 N, kp

Wall mounted crane (3 forces): F_A and F_B to be determin.



k 25 $F_{Ax} = F_B = \frac{a}{b} F_L$; $F_{Ay} = F_L$

STATICS

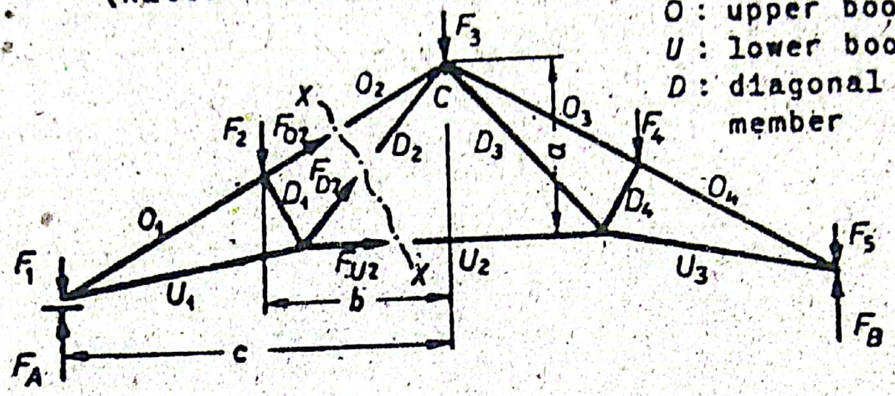
Lattice girder

K 5

Mathematical determination of member loads

(Ritter method - Method of Sections)

O : upper boom
U : lower boom
D : diagonal member



k 26

Determine the support reactions from K 4 (girder on two supports). Draw a line X...X through the framework to the bar in question, but intersecting no more than 3 bars. Take tensile forces as positive, so that compressive forces are negative.

Establish the equation of moments $\sum M = 0$ with moments of external and internal forces taken about the point of intersection of two unknown forces.

Rule for moment signs

Where turning moment is counter-clockwise the sign is positive

Where turning moment is clockwise the sign is negative

Example (from the above girder)

problem: to find force F_{U2} in bar U_2.

solution:

Draw a line X...X through $O_2 - D_2 - U_2$. Since the lines O_2 and D_2 meet at C, this is the relative point of intersection selected so that the moment O_2 and D_2 may equal zero.

Proceed as follows: $\sum M_C = 0$

$$+ a\,F_{U2} + b\,F_2 - c(F_A - F_1) = 0$$

$$F_{U2} = \frac{-\,b\,F_2 + c(F_A - F_1)}{a}$$

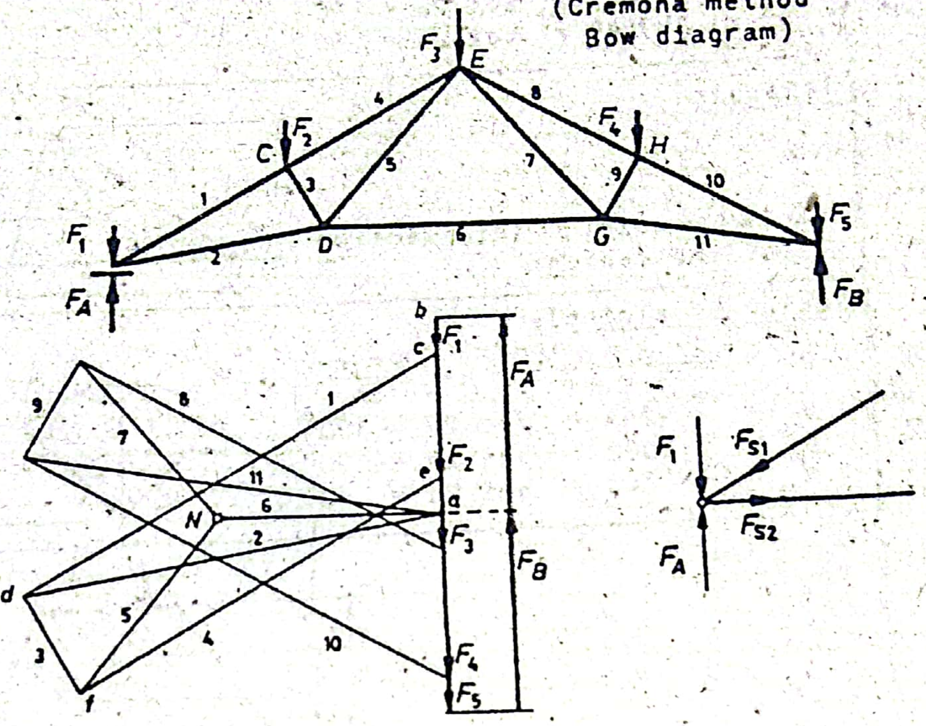
STATICS

Lattice girder

K 6

Graphical determination of forces in members

(Cremona method – Bow diagram)



Basic principles

Each bar is confined by two adjacent joints. The external forces only act through the joints.

Procedure

Establish a scale of forces and determine the support reactions. Since each force polygon must not contain more than two unknown forces, start at joint *A*. Establish identical order of forces (cw or ccw) for all joints (e.g. $F_A - F_1 - F_{S1} - F_{S2}$).

Joint *A*: Force polygon $a - b - c - d - a$. Keep a record of forces being tensile or compressional.

Joint *C*: Force polygon $d - c - e - f - d$.

u.s.w.

Check

Forces acting through a single joint in the framework form a polygon in the Bow diagram.

Forces acting through a single point in the diagram of forces form a triangle in the framework.

Centre of gravity

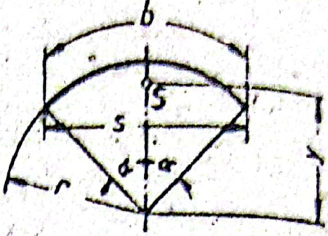
Arc of circle

k 28 $y = \frac{r \sin\alpha \; 180^\circ}{\pi \; \alpha} = \frac{r\, s}{b}$

k 29 $y = 0{\cdot}6366\, r$ bei $2\alpha = 180^\circ$

k 30 $y = 0{\cdot}9003\, r$ bei $2\alpha = 90^\circ$

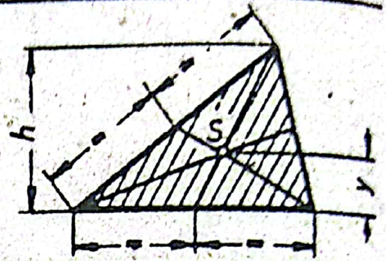
k 31 $y = 0{\cdot}9549\, r$ bei $2\alpha = 60^\circ$



Triangle

k 31 $y = \frac{1}{3} h$

S is the point of intersection of the medians



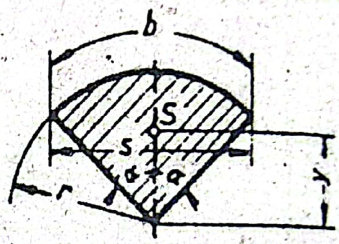
Sector of a circle

k 32 $y = \frac{2r \sin\alpha \; 180^\circ}{3 \; \pi \; \alpha} = \frac{2r\, s}{3b}$

k 33 $y = 0{\cdot}4244\, r$ bei $2\alpha = 180^\circ$

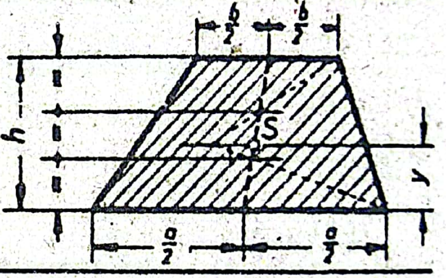
k 34 $y = 0{\cdot}6002\, r$ bei $2\alpha = 90^\circ$

k 35 $y = 0{\cdot}6366\, r$ bei $2\alpha = 60^\circ$



Trapezium

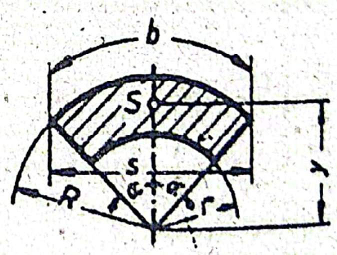
k 36 $y = \frac{h}{3} \times \frac{a + 2b}{a + b}$



Sector of an annulus

k 37 $y = \frac{2}{3} \times \frac{R^3 - r^3}{R^2 - r^2} \times \frac{\sin\alpha}{\operatorname{arc}\alpha}$

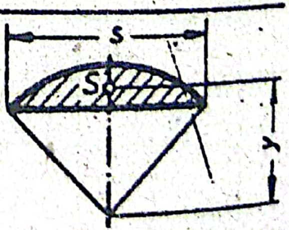
k 38 $= \frac{2}{3} \times \frac{R^3 - r^3}{R^2 - r^2} \times \frac{s}{b}$



Segment of a circle

k 39 $y = \frac{s^3}{12\, A}$

for area A see B 3



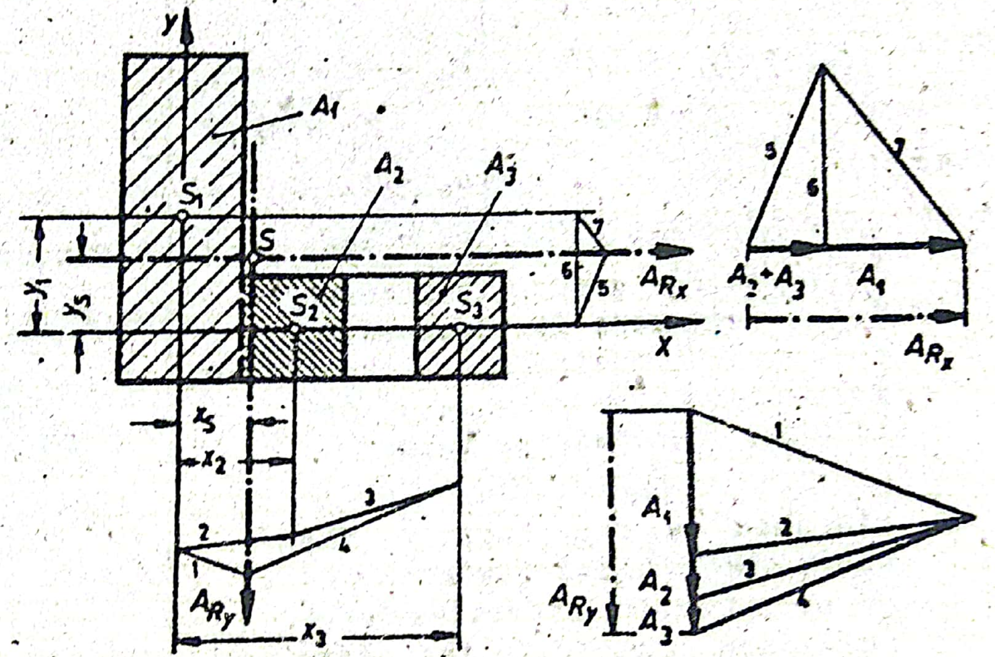
For determination of centre of gravity S, see also J 7

Determination of centre of gravity of any random surface area

Graphical solution

Subdivide the total area A into partial areas A_1, $A_2 \dots A_n$ the centres of gravity of which are known. The size of each partial area is represented as a force applied to the centre of area of each partial area. Use the force polygon (see K 2) to determine the mean forces A_{R_x} and A_{R_y} operating in any two directions (preferably at right angles). The point of intersection of the lines of application will indicate the position of the centre of area A.

k 40



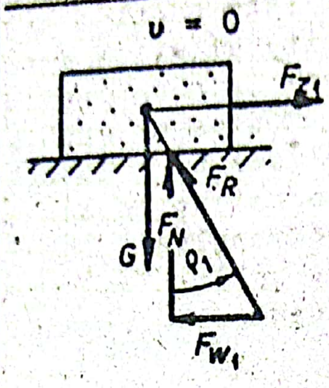
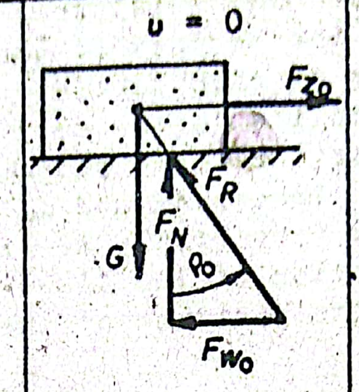
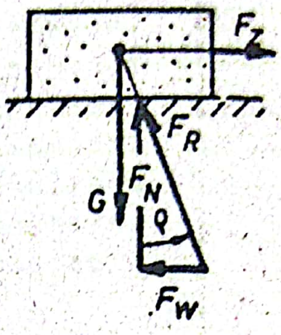
Mathematical solution

Subdivide the above total area A into partial areas A_1, $A_2 \dots A_n$; we now get

	distance	in general	in the above example
k 41	$x_s =$	$\dfrac{\sum_{i=1}^{n} A_i\ x_i}{A}$	$\dfrac{A_1\ x_1 + A_2\ x_2 + A_3\ x_3}{A}$
k 42	$y_s =$	$\dfrac{\sum_{i=1}^{n} A_i\ y_i}{A}$	$\dfrac{A_1\ y_1 + A_2\ y_2 + A_3\ y_3}{A}$

Note: In the above example the distances x_1, y_2 and y_3 each equal zero.

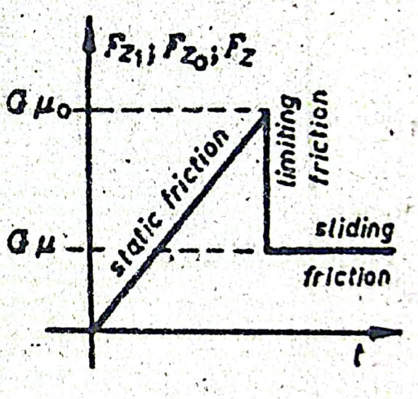
Force acting parallel to a sliding plane

	static friction	limiting frict.	sliding friction
k 43	 $\upsilon = 0$ F_{Z1} F_R F_N G ϱ_1 F_{W1}	 $\upsilon = 0$ F_{Z0} F_R F_N G ϱ_0 F_{W0}	 $\upsilon > 0$ F_Z F_R F_N G ϱ F_W
k 44	$F_{W_1} = -F_{Z_1} = G \tan \varrho_1$	$F_{W_0} = -F_{Z_0} = G \tan \varrho_0$	$F_W = -F_Z = G \tan \varrho$
k 45	$F_N = -G$	$F_N = -G$	$F_N = -G$
k 46		$\mu_0 = \tan \varrho_0 > \mu$	$\mu = \tan \varrho < \mu_0$
k 47	$0 < \varrho_1 (\text{variable}) < \varrho_0$	$\varrho_0 = \text{const.} > \varrho$	$\varrho = \text{const.} < \varrho_0$

A force F_{Z1} gradually increasing from zero is compensated by an increasing F_{W1} without causing the body to move, until F_{Z1} reaches the value

k 48 $$F_{Z0} = G \mu_0 .$$

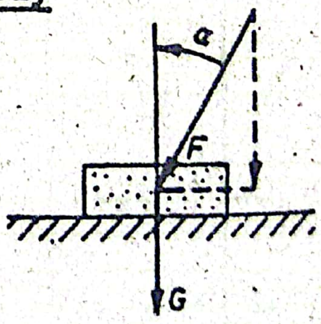
As soon as this happens, the body starts sliding, whereby F_Z drops to $G\mu$. Any excessive force will now accelerate the body.



Force applied obliquely

The force F needed to set in motion a body, weight G:

k 49 $$F = G \frac{\mu_0}{\sin \alpha - \mu_0 \cos \alpha} = G \frac{\sin \varrho_0}{\sin(\alpha - \varrho_0)}$$



The force needed to maintain the motion is ascertained by replacing μ_0 by μ. No motion possible when result of F is negative.

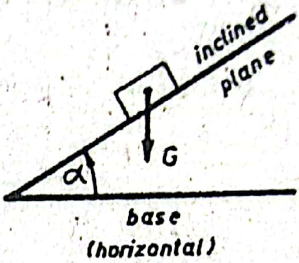
F_{W1}, F_{W0}, F_W : frict. coeff. | –, μ_0, μ: frict. force
F_{Z1}, F_{Z0}, F_Z : angle of frict. | $\varrho_1, \varrho_0, \varrho$: tract. force

Inclined plane

General

The angle α at which a body will move easily down an inclined plane is the angle of friction ϱ.

$$\tan\alpha = \tan\varrho = \mu$$

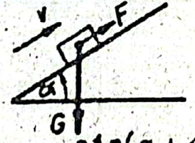
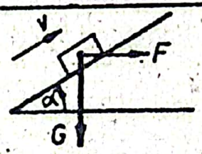
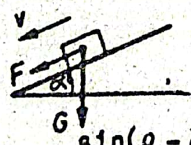
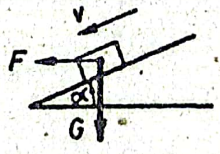
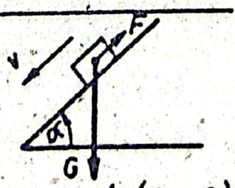
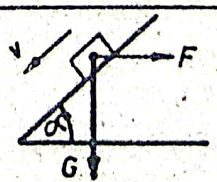


k 50 Application in the experimental determination of the angle of friction or the friction coefficient:

$$\alpha = \tan\varrho$$

k 51 **Condition of automatic locking:** $\alpha < \varrho$

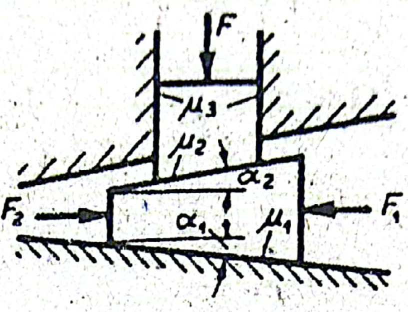
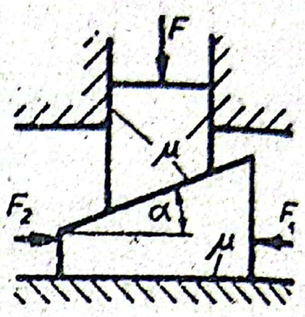
Friction properties

	motion	constant velocity maintained by tractive force F parallel to	
		inclined plane	base
k 52	upwards $0 < \alpha < \alpha^*$	 $F = G\,\dfrac{\sin(\alpha+\varrho)}{\cos\varrho}$	 $F = G\,\tan(\alpha+\varrho)$
k 53	downwards $0 < \alpha < \varrho$	 $F = G\,\dfrac{\sin(\varrho-\alpha)}{\cos\varrho}$	 $F = G\,\tan(\varrho-\alpha)$
k 54	downwards $\varrho < \alpha < \alpha^*$	 $F = G\,\dfrac{\sin(\alpha-\varrho)}{\cos\varrho}$	 $F = G\,\tan(\alpha-\varrho)$

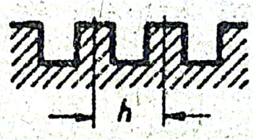
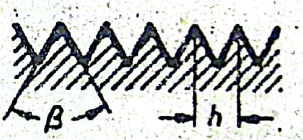
Note: For static friction replace μ by μ_0 and ϱ by ϱ_0.

α^*: Tilting angle of body

Wedges

			
k 55	driving in	$F_1 = F\,\dfrac{\tan(\alpha_1+\varrho_1)+\tan(\alpha_2+\varrho_2)}{1-\tan\varrho_3\,\tan(\alpha_2+\varrho_2)}$	$F_1 = F\,\tan(\alpha+2\varrho)$
k 56	loose-ning	$F_2 = F\,\dfrac{\tan(\alpha_1-\varrho_1)+\tan(\alpha_2-\varrho_2)}{1+\tan\varrho_3\,\tan(\alpha_2-\varrho_2)}$	$F_2 = F\,\tan(\alpha-2\varrho)$
k 57	autom. locking	$\alpha_1+\alpha_2 \leqq \varrho_{0_1}+\varrho_{0_2}$	$\alpha \leqq 2\varrho_0$

Screws

				
k 58	turning moment when	raising	$M_1 = F\,r\,\tan(\alpha+\varrho)$	$M_1 = F\,r\,\tan(\alpha+\varrho')$
k 59		lowering	$M_2 = F\,r\,\tan(\alpha-\varrho)$	$M_2 = F\,r\,\tan(\alpha-\varrho')$
k 60	conditions of automatic locking when lowering		$\alpha < \varrho$	$\alpha < \varrho'$
k 61	efficiency of screw when	raised	$\eta = \dfrac{\tan\alpha}{\tan(\alpha+\varrho)}$	$\eta = \dfrac{\tan\alpha}{\tan(\alpha+\varrho')}$
k 62		lowered	$\eta = \dfrac{\tan\alpha}{\tan(\alpha-\varrho)}$	$\eta = \dfrac{\tan\alpha}{\tan(\alpha-\varrho')}$

M_1 : raising moment N m, kp m

M_2 : lowering moment N m, kp m

α : lead of thread $\left(\tan\alpha = \dfrac{h}{2\,\pi\,r}\right)$

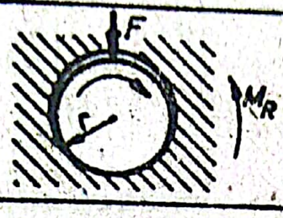
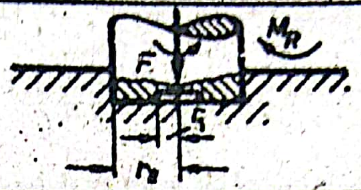
k 63 ϱ : angle of friction $(\tan\varrho = \mu)$

k 64 ϱ' : angle of friction $\left(\tan\varrho' = \dfrac{\mu}{\cos\beta/2}\right)$

k 65 r : mean radius of thread m, mm

STATICS
Friction

Bearing friction

	radial bearing	longitudinal bearing
		
k 66	$M_R = \mu_q\, r\, F$	$M_R = \mu_l \frac{r_1 + r_2}{2} F$

M_R : moment of friction

μ_q : coefficient of radial bearing (not constant values)

μ_l : friction of a longitudinal

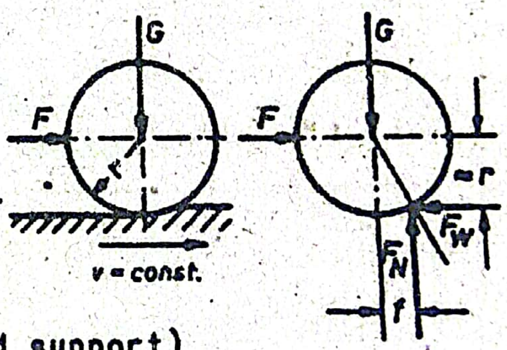
Note: μ_q and μ_l are determined experimentally as a function of bearing condition, bearing clearance,

k 67 and lubrication. For run in condition: $\mu_0 \approx \mu_l \approx \mu_q$.

k 68 Always use $r_1 > 0$ to allow for lubrication.

Rolling resistance

Rolling of a cylinder



k 69 $F = \frac{f}{r} F_N \approx \frac{f}{r} G$

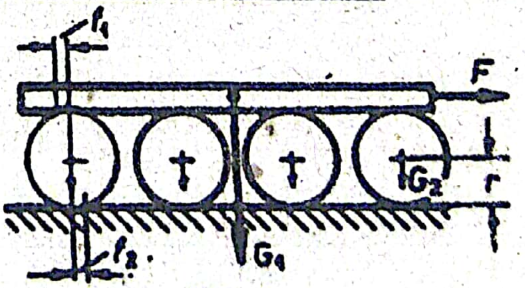
k 70 Rolling condit.: $F_W < \mu_0 F_N$

F_W: force of rolling resist.

f : lever arm of rolling resistance – value on Z 16 (caused by deformation of cylinder and support)

μ_0: coeff. of static frict. between cylinder and support

Displacement of a plate supported by cylinders



k 71 $F = \frac{(f_1 + f_2)\, G_1 + n\, f_2\, G_2}{2\, r}$

k 72 where $f_1 = f_2 = f$ and $n\, G_2 < G_1$:

k 73 $F = \frac{f}{r} G_1$

G_1 , G_2 : weight of plate and one cylinder

f_1 , f_2 : lever arms of force of rolling resistance

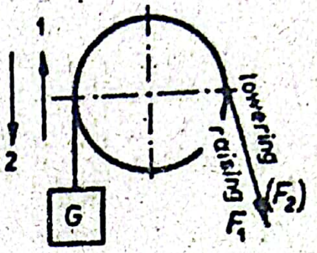
F : tractive force

r : radius of cylinder | n : number of cylinders

Friction

Rope friction

	tract. force and frict. force for raising load	lowering load
k 75	$F_1 = e^{\mu\hat{\alpha}}\,G$	$F_2 = e^{-\mu\hat{\alpha}}\,G$
k 76	$F_R = (e^{\mu\hat{\alpha}} - 1)G$	$F_R = (1 - e^{-\mu\hat{\alpha}})G$



Formulae apply where cylinder is stationary and rope is moving at constant velocity, or where rope is stationary and cylinder is rotating at constant angular velocity.

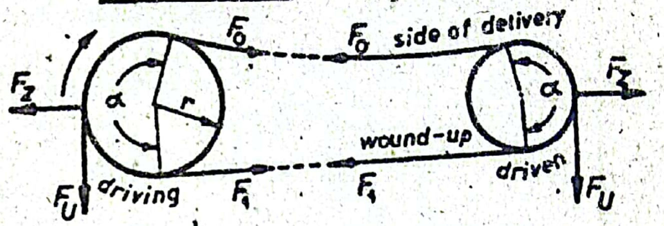
k 77 **Condit. of equilibrium:** $F_2 < F < F_1$ | $G\,e^{-\mu\hat{\alpha}} < F < G\,e^{\mu\hat{\alpha}}$

(F : force without friction)

Belt drive

k 78 $F_U = \dfrac{M_a}{r}$

k 79 $F_U = F_R$

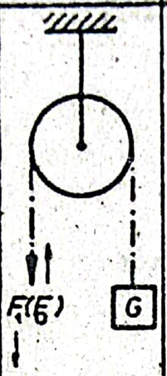
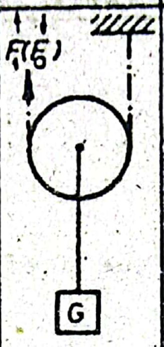
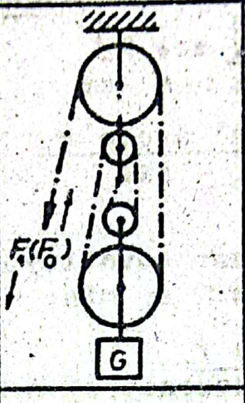
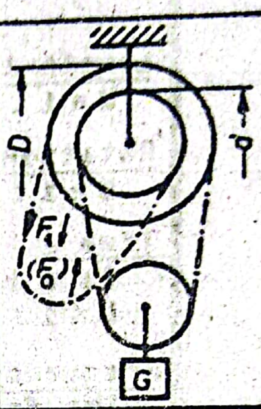


	forces	in motion	at rest
k 80	F_o	$F_o = \dfrac{F_U}{e^{\mu\hat{\alpha}} - 1}$	$F_o = F_1 = \dfrac{F_z}{2}\dfrac{(e^{\mu\hat{\alpha}} + 1)}{(e^{\mu\hat{\alpha}} - 1)}$
k 81	F_1	$F_1 = F_U\dfrac{e^{\mu\hat{\alpha}}}{e^{\mu\hat{\alpha}} - 1}$	
k 82	F_z	$F_z = F_U\dfrac{e^{\mu\hat{\alpha}} + 1}{e^{\mu\hat{\alpha}} - 1}$	

F_U : tangentiel force of driving wheel
F_R : frictional force of rope
M_a : driving torque
$\hat{\alpha}$: angle of contact (radians). Always introduce lowest value into formula
μ : coeff. of sliding frict. (value of exper. obtained for leather belt on steel drum: $\mu = 0{,}22 + 0{,}012\,v$ s/m) k 83
v : belt velocity
e = 2.718 281 83... (base of natural logs)

Rope operated machines

The following figures deal solely with rope rigidity, disregarding bearing friction.

	unknown quantity	fixed sheave	free sheave	pulley block ordinary	pulley block differential
					
k 84	$F_1 =$	$\varepsilon \cdot G$	$\frac{\varepsilon}{1 + \varepsilon} G$	$\frac{\varepsilon^n(\varepsilon - 1)}{\varepsilon^n - 1} G$	$\frac{\varepsilon^2 - \frac{d}{D}}{\varepsilon + 1} G$
k 85	$F_0 =$	$\frac{1}{\varepsilon} G$	$\frac{1}{1 + \varepsilon} G$	$\frac{\frac{1}{\varepsilon^n}\left(\frac{1}{\varepsilon} - 1\right)}{\frac{1}{\varepsilon^n} - 1} G$	$\frac{\varepsilon}{1+\varepsilon}\left(\frac{1}{\varepsilon^2} - \frac{d}{D}\right) G$
k 86	$F =$	G	$\frac{1}{2} G$	$\frac{1}{n} G$	$\frac{1}{2}\left(1 - \frac{d}{D}\right) G$
k 87	$s =$	h	$2\ h$	$n\ h$	$\frac{2}{1 - \frac{d}{D}} h$

k 88 mechanical advantage $= \frac{\text{force}}{\text{effort}} = \frac{F}{G} = \frac{h}{s}$

F_1 : force required to raise load, disregarding
F_0 : force required to lower load, disregarding
F : force, disregarding both rope rigidity and
} bearing friction

k 89 $\varepsilon = \frac{1}{\eta}$: loss factor for rope rigidity (for wire ropes and chains $\approx 1{\cdot}05$)

η : efficiency
n : number of sheaves
h : path of load
s : path of force

KINEMATICS

General terms

General

Kinematics deals with motions of bodies as a function of time.

The most important quantities of kinematics and their units

Length l, see K 1

Units: m; km

Rotational angle φ

Unit: rad

Time t

Is a base quantity, see preface.

Units: s; min; h

Frequency f

The frequency of a harmonic or sinusoidal oscillation is the ratio of the number of periods (full cycles) and the corresponding time.

L 1 $$f = \frac{\text{number of oscillations}}{\text{corresponding time}}$$

Units: Hz (Hertz) = 1/s = cycle/s; 1/min

Period T

The period T is the time required for one full cycle. It is the reciprocal of the frequency f.

L 2 $$T = \frac{1}{f}$$

Units: s; min; h

Rotational speed n

Where an oscillation is tightly coupled with the rotation of a shaft, and one revolution of the shaft corresponds exactly to one full cycle of the oscillation, the rotational speed n of the shaft is equal to the frequency f of the oscillation.

L 3 $$n = f \; ; \quad f = \frac{n\ \mathrm{min}}{60}$$

Units: revolutions/second (1/s)
rev/min, r.p.m. (1/min)

continued on L 2

KINEMATICS

General terms

L 2

continued from L 1

Velocity υ

The velocity υ is the first derivative of the distance s with respect to the time t:

14 $$\upsilon = \frac{ds}{dt} = \dot{s}$$

Where the velocity is constant, the following relation applies:

15 $$\upsilon = \frac{s}{t}$$

Units: m/s; km/h

Angular velocity ω, angular frequency ω

The angular velocity ω is the first derivative of the angle turned through φ, with respect to the time t:

16 $$\omega = \frac{d\varphi}{dt} = \dot{\varphi}$$

Hence, for constant angular velocity:

17 $$\omega = \frac{\varphi}{t}$$

Where $f = n$ (see 13), the angular velocity ω is equal to the angular frequency ω.

18 $$\omega = 2\pi f = 2\pi n = \dot{\varphi}$$

Units: 1/s; rad/s; 1°/s

Acceleration a

The acceleration a is the first derivative of the velocity υ with respect to the time t:

9 $$a = \frac{d\upsilon}{dt} = \dot{\upsilon} = \frac{d^2s}{dt^2} = \ddot{s}$$

Units: m/s²; km/h²

Angular acceleration α

The angular acceleration α is the first derivative of the angular velocity ω with respect to the time t:

110 $$\alpha = \frac{d\omega}{dt} = \dot{\omega} = \frac{d^2\varphi}{dt^2} = \ddot{\varphi}$$

Units: $1/s^2$; rad/s^2; $1°/s^2$

KINEMATICS

General

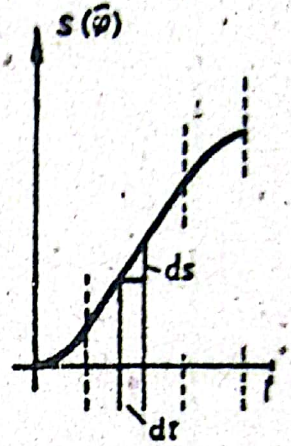
Distance, velocity, and acceleration of mass point in motion

Distance-time curve

An s-t curve is recorded for the motion. The first derivative of this curve is the instantaneous velocity v:

1 11 $$v = \frac{ds}{dt} = \dot{s}$$

It is the slope dt the tangent to the s-t curve.



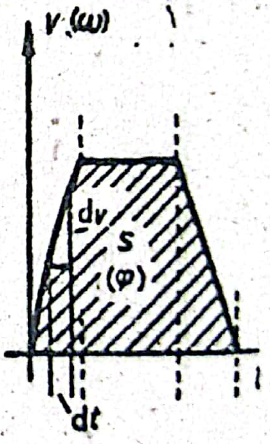
Velocity-time curve

The velocity-time history is expressed as a v-t curve. The first derivative of this curve is the instantaneous acceleration a. Hence the acceleration is the second derivative of the distance-time curve.

1 12 $$a = \frac{dv}{dt} = \dot{v} = \ddot{s}$$

It is the slope dt the tangent to the v-t curve.

The shaded area represents the distance travelled $s(t)$.

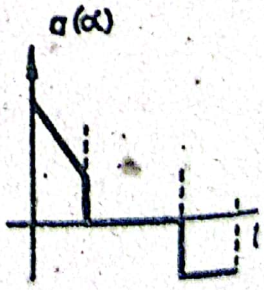


Acceleration-time curve

The acceleration-time history is shown as an a-t curve, which enables peak accelerations to be determined.

1 13 $a > 0$: acceleration (increasing velocity)

1 14 $a < 0$: retardation (decreasing velocity)



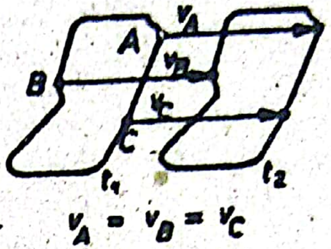
Note to diagrams

The letters in brackets apply to rotations (for e planation see L 3 and L 4).

KINEMATICS

The most important kinds of motion

Linear motion

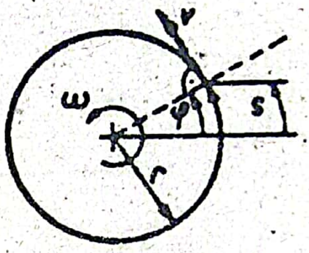


Paths are straight lines. All points of a body cover congruent paths.

Special linear motions

	uniform	uniform accelerated
	motion	
15	$v = v_0 =$ const.	$a = a_0$ = constant

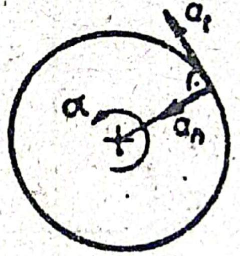
Rotational motion



Paths are circles about the axis. Angle turned through $\widehat{\varphi}$, angular velocity ω, and angular acceleration α are identical for all points of the body.

Special rotational motions

	uniform	uniform accelerated
	motion	
16	$\omega = \omega_0 =$ const.	$\alpha = a_0$ = constant

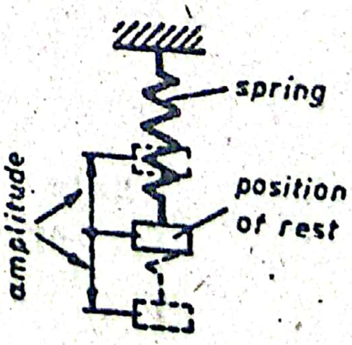


Distance s, velocity v, and tangential acceleration a_t are proportional to the radius:

17 $s = r\widehat{\varphi}$; $v = r\omega$; $a = r\alpha = a_t$

18 centripetal acceleration $a_n = \omega^2 r = \frac{v^2}{r}$

Harmonic oscillation

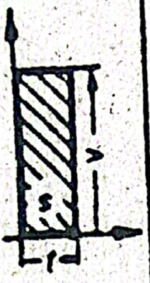
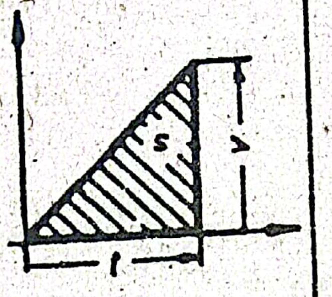
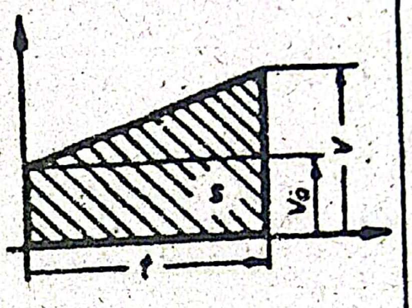


Paths are straight lines or circles. The body moves back and forth about a position of rest. The maximum deflection from this position is called "amplitude".
Instantaneous position, velocity, and acceleration are harmonic functions of time.

KINEMATICS

Linear motion

Uniform and uniform accelerated linear motion

	unknown parameter	uniform $a = 0$ $v = \text{const.}$	uniform { accelerated ($a > 0$) / retarded ($a < 0$) } $a = \text{constant}$ $v_0 = 0$	$v_0 > 0$	EU
					
1 19	$s =$	$v\,t$	$\frac{v\,t}{2} = \frac{a\,t^2}{2} = \frac{v^2}{2\,a}$	$\frac{t}{2}(v_0 + v) = v_0 t + \frac{1}{2} a t^2$	m cm km
1 20	$v =$	$\frac{s}{t}$	$\sqrt{2as} = \frac{2\,s}{t} = a\,t$	$v_0 + a\,t = \sqrt{v_0^2 + 2as}$	m/s cm/s km/h
1 21	$v_0 =$	const.	0	$v - a\,t = \sqrt{v^2 - 2as}$	
1 22	$a =$	0	$\frac{v}{t} = \frac{2s}{t^2} = \frac{v^2}{2s}$	$\frac{v - v_0}{t} = \frac{v^2 - v_0^2}{2s}$	m/s² cm/h² km/h²
1 23	$t =$	$\frac{s}{v}$	$\sqrt{\frac{2\,s}{a}} = \frac{v}{a} = \frac{2\,s}{v}$	$\frac{v - v_0}{a} = \frac{2s}{v_0 + v}$	s min h

Note

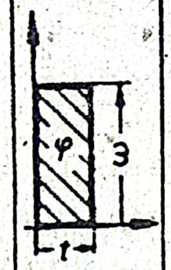
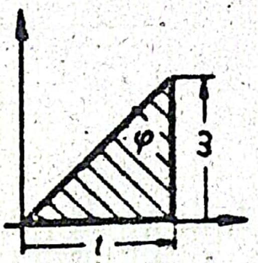
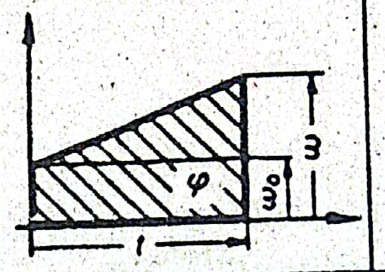
The shaded areas represent the distance s covered during the time period t.

KINEMATICS

Rotational motion

L 6

Uniform and uniform accelerated rotation about a fixed axis

	unknown parameter	uniform	uniform	accelerated $(\alpha > 0)$ retarded $(\alpha < 0)$	EU
		$\alpha = 0$ ω=const.	α = constant $\omega_0 = 0$	$\omega_0 > 0$	
					
1 24	$\varphi =$	ωt	$\frac{\omega t}{2} = \frac{\alpha t^2}{2} = \frac{\omega^2}{2\alpha}$	$\frac{t}{2}(\omega_0 + \omega) = \omega_0 t + \frac{1}{2}\alpha t^2$	– rad
1 25	$\omega =$	$\frac{\varphi}{t}$	$\sqrt{2\alpha\varphi} = \frac{2\varphi}{t} = \alpha t$	$\omega_0 + \alpha t = \sqrt{\omega_0^2 + 2\alpha\varphi}$	1/s m/m s rad/s
1 26	$\omega_0 =$	const.	0	$\omega - \alpha t = \sqrt{\omega^2 - 2\alpha\varphi}$	
1 27	$\alpha =$	0	$\frac{\omega}{t} = \frac{2\varphi}{t^2} = \frac{\omega^2}{2\varphi}$	$\frac{\omega - \omega_0}{t} = \frac{\omega^2 - \omega_0^2}{2\varphi}$	$1/s^2$ $m/m\,s^2$ rad/s^2
1 28	$t =$	$\frac{\varphi}{\omega}$	$\sqrt{\frac{2\varphi}{\alpha}} = \frac{\omega}{\alpha} = \frac{2\varphi}{\omega}$	$\frac{\omega - \omega_0}{\alpha} = \frac{2\varphi}{\omega_0 + \omega}$	s min h

Note

The shaded areas represent the angle of rotation φ covered during a time period t. (Angle of rotation

$\overset{\frown}{\varphi}$ = 2π × number of rotations resp.

φ = 360° × number of rotations)

KINEMATICS

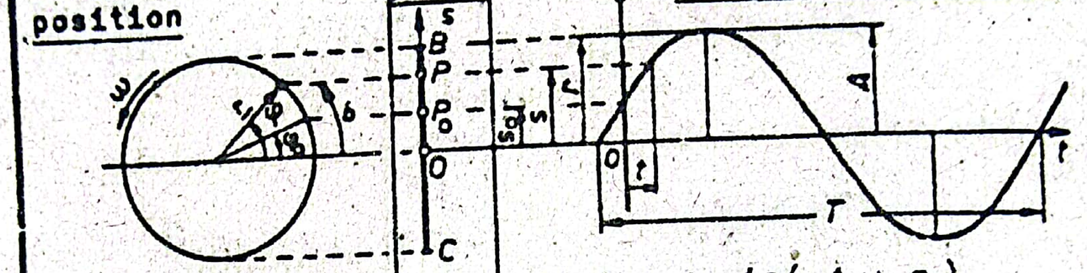
Oscillations

Linear simple harmonic motion

A body supported by a spring performs a linear harmonic oscillation. For this kind of motion, quantities s, v, and a as functions of time are equal to the projections $\underline{s}$, $\underline{v}$, and $\underline{a_n}$ of a uniform rotation of a point.

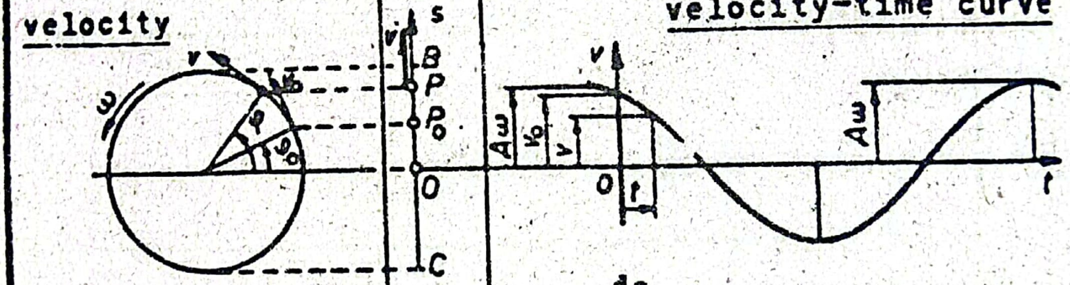
uniform rotation	path (project.)	simple harmonic motion

position — **position-time curve**



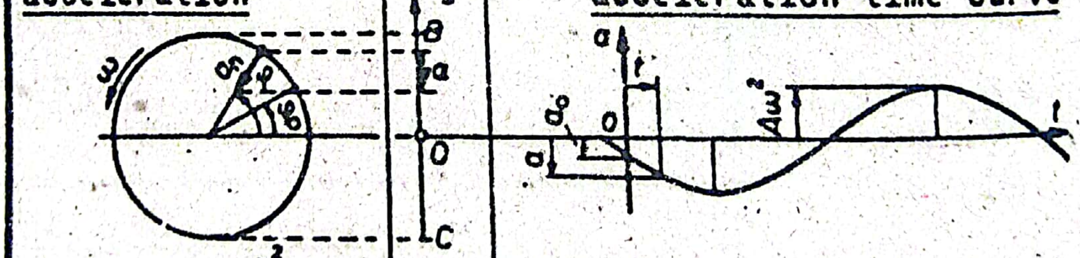
1 29 $\varphi = \omega t + \varphi_0$; $b = r(\omega t + \varphi_0)$ $\qquad$ $s = A \sin(\omega t + \varphi_0)$

velocity — **velocity-time curve**



1 30 $v = r\,\omega$ $\qquad$ $v = \dfrac{ds}{dt} = A\omega \cos(\omega t + \varphi_0)$

acceleration — **acceleration-time curve**



1 31 $a = 0$; $a_n = \dfrac{v^2}{r} = r\omega^2$ $\qquad$ $a = \dfrac{dv}{dt} = -A\omega^2 \sin(\omega t + \varphi_0)$

Differential equation of simple harmonic motion

1 32 $$a = \frac{d^2 s}{dt^2} = -\omega^2 s$$

φ_0 angular position at time $t = 0$
φ angular position at time t
a_n: centripetal accelerat.
$\underline{r}$: radius vector (origin: centre of circle; head: pos. of [body)
B, C: extreme positions of oscillating point

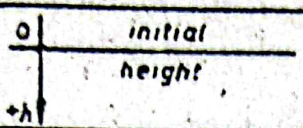
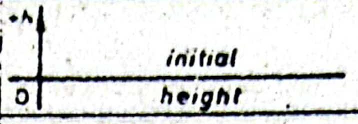
s: displacement
A: amplitude (max. displacem.)
r: radius of circle

KINEMATICS

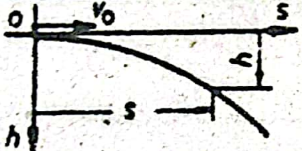
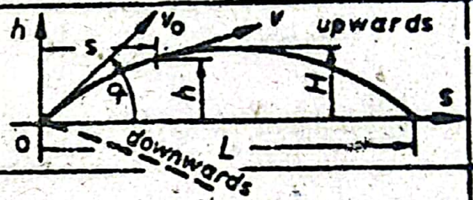
Free fall and projection

L 8

Free fall and vertical projection

	unknown param.	free fall $v_0 = 0$	vertic. project. upwards ($v_0 > 0$) downwds. ($v_0 < 0$)	EU
				
1 33	$h =$	$\frac{g}{2}t^2 = \frac{v}{2}t = \frac{v^2}{2g}$	$v_0 t - \frac{g}{2}t^2 = \frac{v_0 + v}{2}t$	m cm
1 34	$v =$	$g t = \frac{2h}{t} = \sqrt{2gh}$	$v_0 - g t = \sqrt{v_0^2 - 2gh}$	m/s km/h
1 35	$t =$	$\frac{v}{g} = \frac{2h}{v} = \sqrt{\frac{2h}{g}}$	$\frac{v_0 - v}{g} = \frac{2h}{v_0 + v}$	s min

Horizontal and angled projection

	unknown param.	horizont. proj. $v_0 > 0$	angled proj. upwards ($\alpha > 0$) downwds ($\alpha < 0$) $v_0 > 0$	EU
			 h v_0 v upwards s α h H s 0 downwards L	
1 36	$s =$	$v_0 t = v_0\sqrt{\frac{2h}{g}}$	$v_0 t \cos\alpha$	m cm
1 37	$h =$	$\frac{g}{2}t^2$	$v_0 t \sin\alpha - \frac{g}{2}t^2$	m cm
1 38	$v =$	$\sqrt{v_0^2 + g^2 t^2}$	$\sqrt{v_0^2 - 2gh}$	m/s km/s

Range L and max. height H for an angled upwards project.

1 39	general	$L = \frac{v_0^2}{g}\sin 2\alpha$	$H = \frac{v_0^2}{2g}\sin^2\alpha$	m cm
1 40		$t_L = \frac{2v_0}{g}\sin\alpha$	$t_H = \frac{v_0}{g}\sin\alpha$	s min
	maxima	at $\alpha = 45^0$	at $\alpha = 90^0$	
1 41		$L_{max} = \frac{v_0^2}{g}$	$H_{max} = \frac{v_0^2}{2g}$	m cm
1 42		$t_{L_{max}} = \frac{v_0\sqrt{2}}{g}$	$t_{H_{max}} = \frac{v_0}{g}$	s min

α : angle of projection with respect to horizontal plane

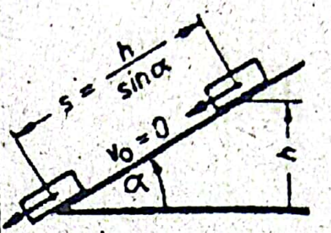
t_H : time for height H | t_L : time for distance L

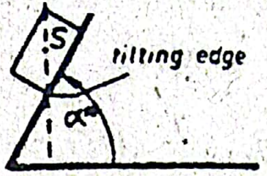
KINEMATICS

Motion on an inclined plane

Sliding motion on an inclined plane

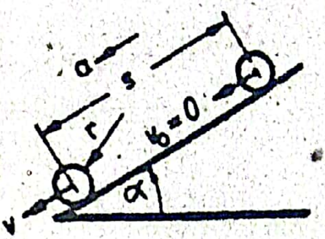
	un-known param.	excluding friction $\mu = 0$	including friction $\mu > 0$
1 43 1 44	$a =$	$g \sin \alpha$	$g(\sin \alpha - \mu \cos \alpha)$ otherwise $g \dfrac{\sin(\alpha - \varrho)}{\cos \varrho}$
1 45	$v =$	$a\, t = \dfrac{2s}{t} = \sqrt{2as}$	
1 46	$s =$	$\dfrac{at^2}{2} = \dfrac{vt}{2} = \dfrac{v^2}{2a}$	
	α	$0 \ldots \alpha^*$	$\varrho_0 \ldots \alpha^*$





Rolling motion on an inclined plane

	un-known param.	excluding friction $f = 0$	including friction $f > 0$
1 47	$a =$	$\dfrac{g\, r^2}{r^2 + k^2} \sin \alpha$	$g r^2 \dfrac{\sin \alpha - \dfrac{f}{r} \cos \alpha}{r^2 + k^2}$
1 48	$v =$	see above 1 45	
1 49	$s =$	see above 1 46	
1 50 1 51	α	$0 \ldots \alpha_{max}$ $\tan \alpha = \mu_0 \dfrac{r^2 + k^2}{k^2}$	α_{min} : $\tan \alpha_{min} = \dfrac{f}{r}$ α_{max} : $\tan \alpha_{max} = \mu_0 \dfrac{r^2 + k^2 + f\, r}{k^2}$



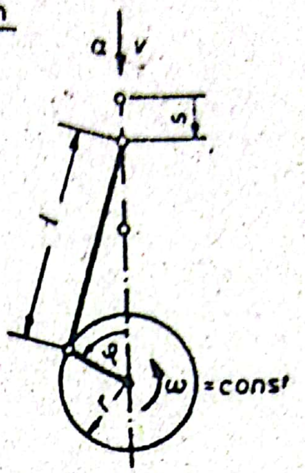
	ball	solid cylinder	pipe with low wall thickness
1 52	$k^2 = \dfrac{2}{5} r^2$	$k^2 = \dfrac{r^2}{2}$	$k^2 = \dfrac{r_1^2 + r_2^2}{2} \approx r^2$

α^* : tilting angle, where centre of gravity S vertically above tilting edge
μ : coefficient of sliding friction (see Z 16)
μ_0 : coefficient of static friction (see Z 16)
1 53 ϱ : angle of sliding friction ($\mu = \tan \varrho$)
1 54 ϱ_0 : angle of static friction ($\mu_0 = \tan \varrho_0$)
f : lever arm of rolling resistance (see k 69 and Z 16)
k : radius of gyration

KINEMATICS

Mechanisms

Slider crank chain



1 55 $s = r(1 - \cos\varphi) + \frac{\lambda}{2} r \sin^2\varphi$

1 56 $v = \omega r \sin\varphi(1 + \lambda \cos\varphi)$

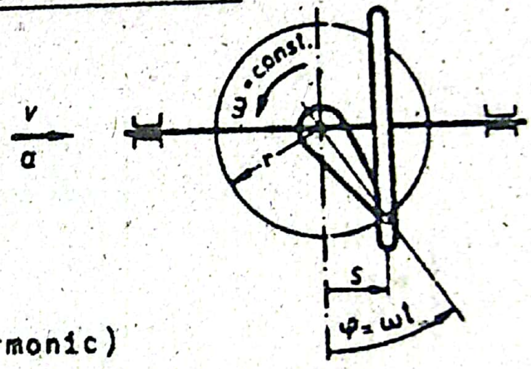
1 57 $a = \omega^2 r (\cos\varphi + \lambda \cos 2\varphi)$

1 58 $\lambda = \frac{r}{l} = \frac{1}{4} \dots \frac{1}{6}$

1 59 $\varphi = \omega t = 2\pi n t$

(λ is called the crank ratio)

Scotch-Yoke mechanism



1 60 $s = r \sin(\omega t)$

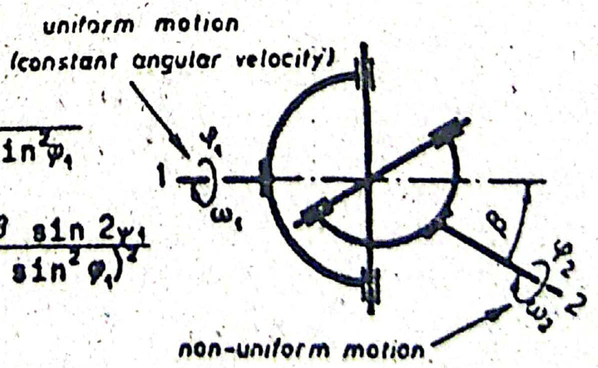
1 61 $v = \omega r \cos(\omega t)$

1 62 $a = -\omega^2 r \sin(\omega t)$

1 63 $\omega = 2\pi n$

(motion is simple harmonic)

Hooke's (Cardan) joint



1 64 $\tan\varphi_2 = \tan\varphi_1 \cdot \cos\beta$

1 65 $\omega_2 = \omega_1 \frac{\cos\beta}{1 - \sin^2\beta \ \sin^2\varphi_1}$

1 66 $\alpha_2 = \omega_1^2 \frac{\sin^2\beta \ \cos\beta \ \sin 2\varphi_1}{(1 - \sin^2\beta \ \sin^2\varphi_1)^2}$

The inclusion of a third shaft parallel to the first provides a uniform drive (constant angular velocity) to the third shaft.

DYNAMICS

General terms

General

Dynamics deals with the forces acting on bodies in motion and with the terms "work, energy, and power".

The most important quantities of dynamics and their units

Mass m (is a base quantity, see preface)

Units: kg; Mg = t; g

1 kg is the mass of the international standard. A mass is measured by means of a steelyard.

Force (gravitational force) F

The force F is the product of mass m and linear acceleration a:

m 1 $$F = m a$$

The gravitational force W is the force acting on a mass m due to the earth's acceleration g:

m 2 $$W = m g$$

Being a gravitational force the weight W is measured by means of a spring balance.

Units: N; kgf; lbf

1 N is the force that, when acting on a body of a mass m = 1 kg for 1 s, accelerates this body to a final velocity of 1 m s^{-1} (i.e. accelerates this body at 1 m s^{-2}). 1 kgf = 9·81 N is the gravitational force acting on a mass of 1 kg due to the earth's attraction.

Work W

The mech. work is the product of force F and distance s, where the constant force F acts on a body in linear motion in a direction parall. to the distance s covered. ($W = F s$)

Units: N m = Joule = J = W s; kgf m; ft lbf

Where a force of 1 N acts over a distance of 1 m, it produces the work (energy) of 1 N m (J).

Power P

The power P is the derivative of work with respect to time. Where work (energy) increases or decreases linearily with time, power is the quotient of work and time. ($P = W/t$)

Units: W(Watt); kgf m s^{-1}; H.P.

Where for a period of 1 s an energy of 1 J is converted at a constant rate, the corresponding power is 1 W.

$$1\ \mathrm{W} = 1\ \mathrm{J/s}$$

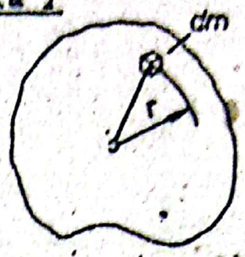
DYNAMICS

Mass, Mass moment of inertia

M 2

Definition of the mass moment of inertia I

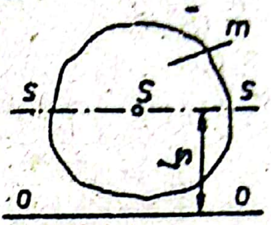
The mass mom. of inertia of a body about an axis has been defined as the sum of the products of mass-elements and the squares of their dist. from the axis.



m 3 $I = \sum r^2 \Delta m = \int r^2 \, dm$ kg m²

Steiner's theorem (Parallel axis theorem) (see also J 9)

Where the mass moment of inertia of a body of mass m about an axis through its centre of gravity S–S is I_{SS}, the mass moment of inertia about a parallel axis 0–0 at a distance l_S will be:



m 4 $I_{OO} = I_{SS} + m \, l_S^2$ kg m²

Radius of gyration k

The radius of gyration of a body of mass m and mass moment of inertia I is the radius of an imaginary cylinder of infinitely small wall thickness having the same mass and the same mass moment of inertia as the body in question.

m 5 hence $k = \sqrt{\frac{I}{m}}$ m, cm, mm

Flywheel effect

m 6 Flywheel effect $W k^2 = m g k^2 = g I$ N m², kg cm³ s⁻²
(k^2 formulae see M 3)

Equivalent mass (for rolling bodies)

m 8 $m_{eq} = \frac{I}{k^2}$ kg

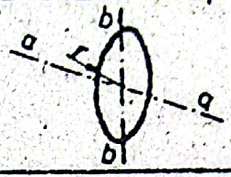
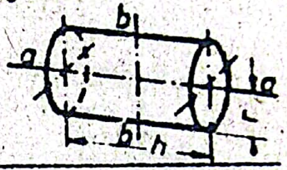
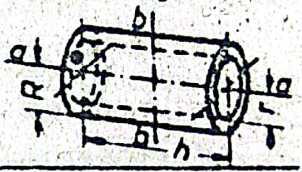
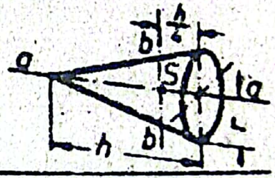
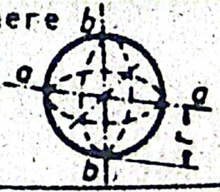
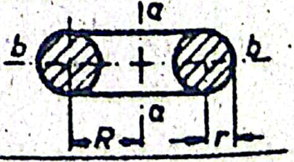
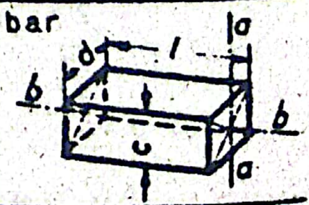
Basic formulae

	linear motion formulae	units	rotational motion formulae	units
m 9	$F_a = m a$	N, kgf	$M_a = I \alpha$	N m, kgf m
m 10	$W = F s$ (F=const.)	N m, kgf m	$W = M \varphi$ (M=const.)	N m, kgf m
m 11	$W_K = \frac{1}{2} m v^2$	J, "	$W_K = \frac{1}{2} I \omega^2$	J, "
m 12	$W_p = m g h$	J, "	$\omega = 2 \pi n$	s⁻¹, min⁻¹
m 13	$W_F = \frac{1}{2} F \Delta l$	J, "	$W_F = \frac{1}{2} M \Delta \beta$	W s, kgf m
m 14	$P = \frac{dW}{dt} = F v$	W, kW	$P = \frac{dW}{dt} = M \omega$	W, kW

For explanation of symbols see M 4

DYNAMICS

Mass, Mass moment of inertia

	Mass moment of inertia about		
	axis a–a (turning axis)	axis b–b passing through centre of gravity S	type of body
m 15	$I = m r^2$	$I = \frac{1}{2} m r^2$	circular hoop 
m 16	$k^2 = r^2$	$k^2 = \frac{1}{2} r^2$	
m 17	$I = \frac{1}{2} m r^2$	$I = \frac{m}{12} (3 r^2 + h^2)$	cylinder 
m 18	$k^2 = \frac{1}{2} r^2$	$k^2 = \frac{1}{12} (3r^2 + h^2)$	
m 19	$I = \frac{1}{2} m(R^2 + r^2)$	$I = \frac{m}{12}(3R^2+3r^2+h^2)$	hollow cylinder 
m 20	$k^2 = \frac{1}{2} (R^2 + r^2)$	$k^2 = \frac{1}{12}(3R^2+3r^2+h^2)$	
m 21	$I = \frac{3}{10} m r^2$	$I = \frac{3}{80} m(4r^2 + h^2)$	cone 
m 22	$k^2 = \frac{3}{10} r^2$	$k^2 = \frac{3}{80} (4r^2 + h^2)$	
m 23	$I = \frac{2}{5} m r^2$	$I = \frac{2}{5} m r^2$	sphere 
m 24	$k^2 = \frac{2}{5} r^2$	$k^2 = \frac{2}{5} r^2$	
m 25	$I = m(R^2+\frac{3}{4} r^2)$	$I = m \frac{4R^2 + 5r^2}{8}$	torus 
m 26	$k^2 = R^2 + \frac{3}{4} r^2$	$k^2 = \frac{1}{8}(4R^2 + 5r^2)$	
m 27	$I = \frac{1}{3} m l^2$	$I = \frac{m}{12} (d^2 + c^2)$	bar 
m 28	$k^2 = \frac{1}{3} l^2$	$k^2 = \frac{1}{12} (d^2 + c^2)$	

DYNAMICS

Rotation

Total kinetic energy of a body

m 29 $$W_K = \frac{1}{2} m v_s^2 + \frac{1}{2} I_s \omega^2 \qquad \text{J, kgf m}$$

Kinetic energy of a rolling body – no sliding

m 30 $$W_K = \frac{1}{2} (m + m_{eq}) v_s^2 \qquad \text{J, kgf m}$$

m 31 $$v_s = \omega r \qquad \text{m/s, km/h}$$

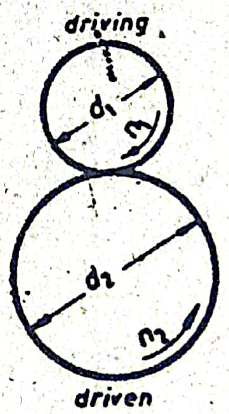
Rotational torque

m 32 $$M = \frac{P}{\omega} = \frac{P}{2 \pi n} \qquad \text{N m, kgf m}$$

Transmission ratios

Transmission ratio

m 33 $$i = \frac{d_2}{d_1} = \frac{z_2}{z_1} = \frac{n_1}{n_2} = \frac{\omega_1}{\omega_2}$$



Torque ratio

m 34 $$\frac{\text{moment of force}}{\text{moment of load}} = \frac{M_F}{M_L} = \frac{1}{i \eta}$$

Efficiency

m 35 $$\eta = \frac{\text{work produced}}{\text{work applied}} = \frac{\text{output}}{\text{input}}$$

Overall efficiency for a series of transmissions

m 36 $$\eta = \eta_1 \times \eta_2 \times \eta_3 \times \ldots$$

m_{eq} : see m 8

v_s : velocity of linear motion of centre of gravity

F_a : accelerating force N, kgf

M_a : accelerating moment N m, kgf m

W_K : kinetic energy J, kgf m

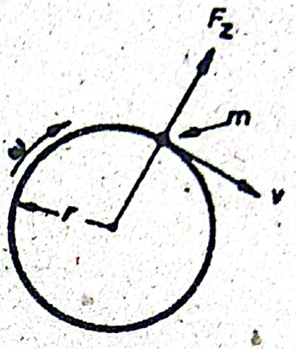
W_P : potential energy J, kgf m

W_F : energy of helical spring under tension J, kgf m

Δl : extension of helical spring

$\Delta \beta$: angular defection of spiral spring (in radians)

Centrifugal force



m 37 $F_z = m\omega^2 r = \frac{m v^2}{r}$ N, kgf

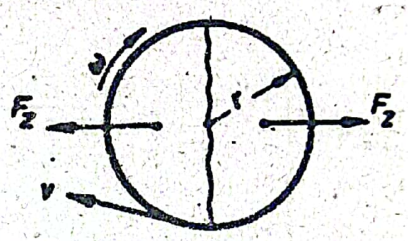
m 38 $= 4\pi^2 m n^2 r$ N, kgf

m 39 $v = 2\pi r n$ m/s, km/h

m 40 $\omega = 2\pi n$ 1/s, 1/min

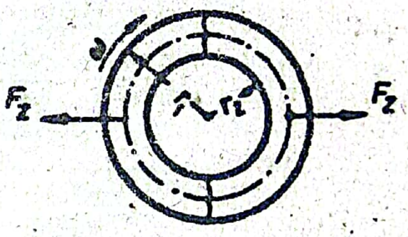
Stresses in rotation bodies

Disc



m 41 $\sigma_z = \frac{\omega^2 r^2 \varrho}{3} = \frac{v^2 \varrho}{3}$ N/m², kgf/cm²

Ring



m 42 $\sigma_z = \frac{\omega^2 \varrho}{3}(r_1^2 + r_1 r_2 + r_2^2)$ N/m², kgf/cm²

---

l_s :	distance from centre of gravity	m, cm, mm
a :	maximum pendulum swing	m, cm, mm
s :	instantaneous pendulum swing	m, cm, mm
F_z :	centrifugal force	N, kgf, gf
I_0 :	mass moment of inertia about O	kg m², kgf m s²
I_s :	mass moment of inertia about S	kg m², kgf m s²
M_t :	moment required to deflect spiral spring by 1 rad = 57·3°	N m, kgf cm
σ_z:	tensile stress	N/m², kgf/cm², kgf/mm²
T :	period of oscillation (B to B' and back)	s, min
v_E :	velocity at E	m/s, cm/s, km/h
v_F :	velocity at F	m/s, cm/s, km/h
W_{KE}:	kinetic energy at E	N m, kgf m

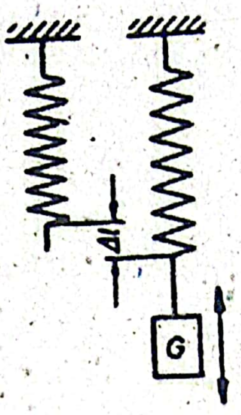
DYNAMICS

Harmonic oscillations

M 6

Mechanical oscillation

General



m 43 period $T = 2\pi\sqrt{\frac{m}{k}}$ s, min

m 44 stiffness $k = \frac{G}{\Delta l}$ N/m, kgf/cm

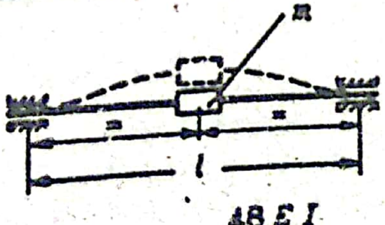
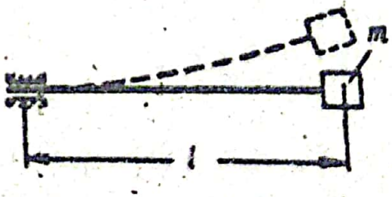
m 45 frequency $f = \frac{1}{T}$ (see L 1) s^{-1}, min^{-1}

m 46 angular velocity $\omega = 2\pi f = \sqrt{\frac{k}{m}}$ s^{-1}, min^{-1}

Critical speed n_c of shaft

m 48 $n_c = \frac{1}{2\pi}\sqrt{\frac{k_q}{m}}$ min^{-1}

stiffness k_q of

two bearing shaft	overhung (cantilever) shaft
	
$k_q = \frac{48 E I}{l^3}$	$k_q = \frac{3 E I}{l^3}$

m 49

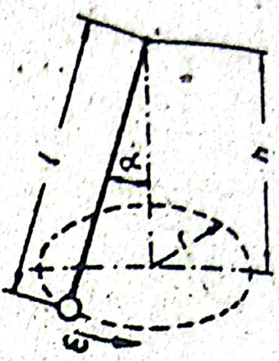
Δl: deflection or elongation
I : second moment of area of shaft cross section
m : mass. When calculating the critical speed the mass m (e.g. of a belt disc) is assumed to be concentrated at a single point. The mass of the shaft should be allowed for by a slight increase
k_q: stiffn [illegible] ansverse oscillations

DYNAMICS

Harmonic oscillations

Pendulum

Conical pendulum



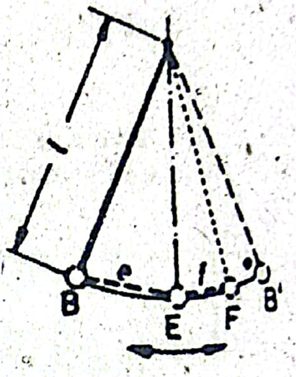
m 50 $T = 2\pi\sqrt{\frac{h}{g}} = 2\pi\sqrt{\frac{l\cos\alpha}{g}}$ s, min

m 51 $\tan\alpha = \frac{r\omega^2}{g} = \frac{r}{h}$

m 52 $\omega = \sqrt{\frac{g}{h}} \quad \Big| \quad h = \frac{g}{\omega^2}$ m, cm

Simple pendulum

The arm of a pendulum has no mass, the total mass is represented as a point.

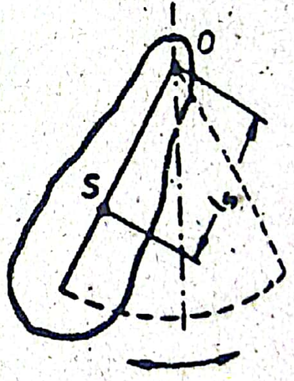


m 53 $T = 2\pi\sqrt{\frac{l}{g}}$ s, min

m 54 $v_E = e\sqrt{\frac{g}{l}} \quad \Big| \quad v_F = \sqrt{\frac{g}{l}(e^2 - f^2)}$ m/s, km/h

m 55 $W_{KE} = m g \frac{e^2}{2 l}$ J, N m, kgf cm

Compound pendulum



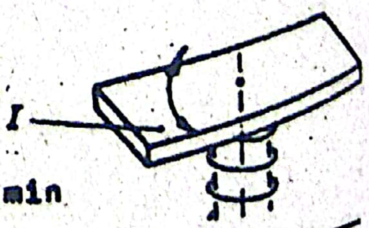
m 56 $T = 2\pi\sqrt{\frac{I_0}{m g l_s}} = 2\pi\sqrt{\frac{k_0^2}{g}}$ s, min

m 57 $I_0 = I_s + m l_s^2$ N m s², kgf cm s²

m 58 $I_s = m g l_s \left(\frac{T^2}{4\pi^2} - \frac{l_s}{g}\right)$ N m s², kgf cm s²

If a body with centre of gravity S is suspended from O, distant l_s from S, and the period of swing determined the mass moment of inertia about O can be calculated using m 58.

Torsional pendulum



m 59 $T = 2\pi\sqrt{\frac{I}{K_1}}$ s, min

For explanation of symbols see M 5

Impact

m 60 When two bodies of mass m_1 and m_2 and velocities v_{11} and v_{21} collide, the total momentum $p = m v$ will remain constant over the whole impact period (velocities become v_{12} and v_{22}):

m 61 $$p = m_1 v_{11} + m_2 v_{21} = m_1 v_{12} + m_2 v_{22}$$

Impact-direction

direct and concentric impact	velocities parallel to normal to surfaces at point of impact	normal to surfaces at point of impact through centre of gravity of both bodies
oblique and concentric impact	any random velocities	
oblique and excentric impact		any random normal to surfaces at point of impact

Types of impact

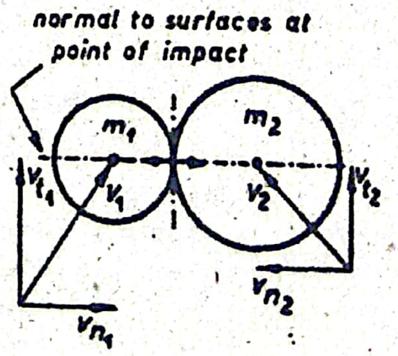
	elastic impact[+)]	plastic impact
relative velocity	equal, before and after impact	equals zero after impact
m 62 m 63 m 64 velocity after direct and concentric impact	$v_{12}=\dfrac{v_{11}(m_1-m_2)+2m_2 v_{21}}{m_1+m_2}$ $v_{22}=\dfrac{v_{21}(m_2-m_1)+2m_1 v_{11}}{m_1+m_2}$	$v_{02}=\dfrac{m_1 v_{11}+m_2 v_{21}}{m_1+m_2}$
m 65 coeff. of restit.	$\varepsilon = 1$	$\varepsilon = 0$

Coefficient of restitution ε

This indicates by what factor the relative velocities will vary before (v_{r1}) and after (v_{r2}) impact:

m 66 $$\varepsilon = \frac{v_{r2}}{v_{r1}}, \qquad \text{here} \qquad 0 \leqq \varepsilon \leqq 1$$

+) For an oblique, concentric, elastic impact the velocity vector v is split into a normal and a tangential component. The normal component v_n produces a direct impact (see above), the tangential component v_t has nu effect on the impact.



HYDRAULICS

General terms

N1

General

Hydraulics deals with two main subjects:

Hydrostatics : The theory of equilibrium in fluids at rest.

Hydrodynamics: The theory of the motion of fluids under the influence of force.

Units of size and terms

Density ϱ

The ratio of mass m to volume V, therefore not subject to the force of gravity and thus an absolute constant for all fluids.

n 1 $$\varrho = \frac{m}{V} \qquad \mathrm{kg/m^3,\ g/mm^3}$$

n 2 for water $\varrho = 1000\ \mathrm{kg/m^3}$

Dynamic viscosity η

Is a material constant of a fluid and is determined experimentally.

Units : Pascalsecond (Pa s); $\mathrm{dPa\,s = dN\,s/m^2}$

n 3 $\mathrm{1\,Pa\,s = 1\,N\,s\,m^2 = 1\,kg/(m\,s)}$

n 4 previously: $\mathrm{1\,kgf\,s/m^2 = 98{\cdot}1\,P}$ (Poise)

$\mathrm{1\,P = 0{\cdot}0102\,kgf\,s/m^2}$

Kinematic viscosity ν

Is the quotient of dynamic viscosity η and density ϱ.

n 5 $$\nu = \frac{\eta}{\varrho}$$

Units : $\mathrm{m^2/s;\ cm^2/s;\ mm^2/s}$

n 6 $\mathrm{1\,Pa\,s\,m^3/kg = 1\,m^2/s}$

n 7 previously: 1St(Stokes)= $\mathrm{1\,cm^2/s = 10^{-4}\,m^2/s = 10^2\,mm^2/s}$

n 8 for water (20°C) $\nu = \mathrm{0{\cdot}01\,St = 10^{-6}\,m^2/s}$

Conversion of Engler degrees (°E) into $\mathrm{m^2/s}$

°E	1·12	1·31	1·48	1·83	2·32	2·87	4·07	5·33	6·67	7·93
$\mathrm{10^{-6}m^2/s}$	2	4	6	10	15	20	30	40	50	60

For values exceeding $\mathrm{60 \times 10^{-6}m^2/s}$ the conversion rate is constant.

Pressure p

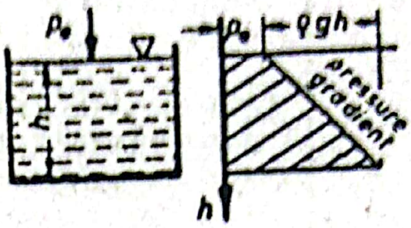
Pressure is the ratio of force and area the force is applied to:

$$p = \frac{F}{A} \qquad \mathrm{N/m^2,\ kgf/cm^2}$$

Pressure in a fluid

At a depth h the pressure is:

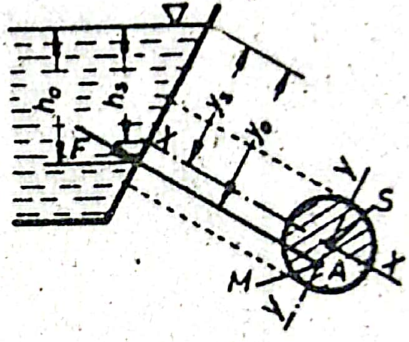
n 10 $$p = p_0 + \varrho g h$$



Forces acting on plane surfaces

Total force on a surface A

n 11 $$F = \varrho g h_s A_v \quad \text{N, kgf}$$

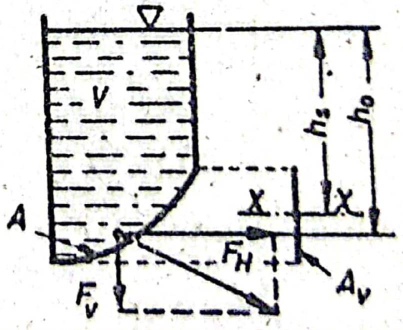


distance from point of application for total pressure at A for

vertical	inclined
walls	
n 12 $h_0 = h_s + \dfrac{I_s}{h_s A}$	$y_0 = y_s + \dfrac{I_s}{y_s A}$

Forces acting on curved surfaces

force	
vertical	horizontal
n 13 $F_v = \varrho g V$	$F_H = \varrho g h_s A_v$
	n 14 $h_0 = h_s + \dfrac{I_{sv}}{h_s A_v}$



ϱ : density of fluid
h_s : distance from centre of area to free surface of fluid
A_v : vertical projection of surface A
S : centre of area of surface
I_s, or I_{sv} : second moment of area of surface A or A_v in relation to axis passing through the centre of area $X...X$ (see P 3), presupposing this to be symmetrical with $Y...Y$.

HYDRAULICS

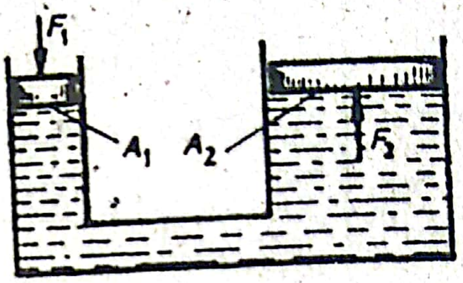
Hydrostatics

Pressure in a compressed fluid
(Pascal's theorem)

Pressure is always constant in any fluid (fluids are incompressible thus their density will remain constant under varying pressures).

Hydrostatic pressure

n 15 $p = \frac{F_1}{A_1} = \frac{F_2}{A_2}$



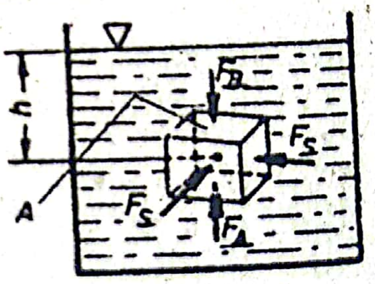
Piston forces

n 16 $F_1 = p A_1 = F_2 \frac{A_1}{A_2}$

17 $F_2 = p A_2 = F_1 \frac{A_2}{A_1}$

Gravitational pressure

Pressure at a mean height h on a small cube with surfaces A is identical on all its sides.



n 18 $F_B = F_S = F_A = \rho g h A$

Buoyancy
(Archimedes principle)

The buoyancy F_A of a body is equal to the weight of water displaced.

n 19 $F_A = \rho g V$

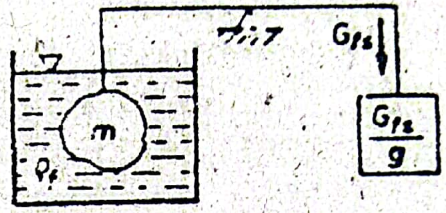
where F_A is	less than	the weight of the body it will	sink
	equal to		remain suspended
	greater than		float

CamScanner

Determination of density ϱ of bodies

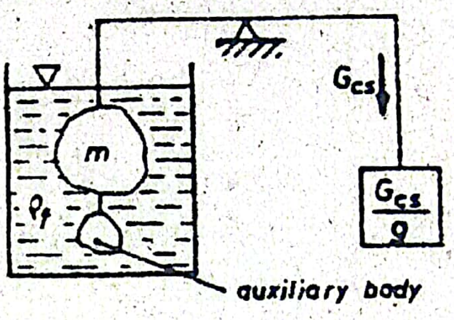
Solid bodies heavier than the fluid used

n 20 $$\varrho = \varrho_f \frac{m}{m - \frac{G_{fs}}{g}}$$



Solid bodies lighter than the fluid used

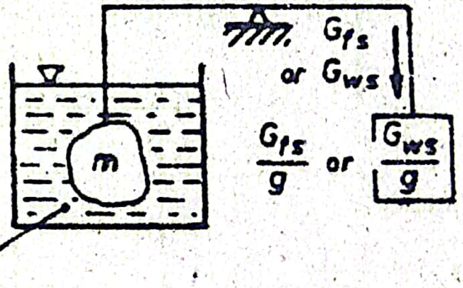
n 21 $$\varrho = \frac{\varrho_f\, m}{m + \frac{G_{Hs}}{g} - \frac{G_{cs}}{g}}$$



Fluids

For any arbitrary solid body determine G_{fs}, G_{ws}, and m. Then the density ϱ_f of the fluid is:

n 22 $$\varrho = \frac{m - \frac{G_{fs}}{g}}{m - \frac{G_{ws}}{g}}\, \varrho_w$$



G_{cs} : gravitational force of the two bodies suspended in the fluid
G_{fs} : gravitational force of body suspended in the fluid
G_{Hs} : gravitational force of auxiliary body in the fluid
G_{ws} : gravitational force of body suspended in the water
m : mass of the body
ϱ_w : density of water
ϱ_f : density of the fluid applied

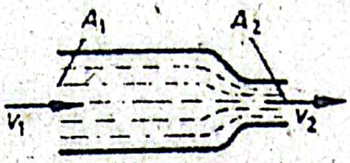
HYDRAULICS

Hydrodynamics

N 5

Flow equation

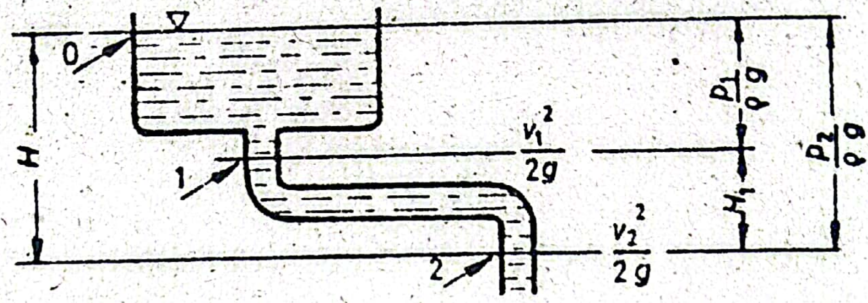
Velocities υ are in inverse proportion to the cross section A through which they flow.



n 23 volume flow $\dot{V} = \upsilon_1 A_1 = \upsilon_2 A_2 = \ldots$ m^3/s, l/min

n 24 mass flow $\dot{m} = \upsilon_1 A_1 \varrho = \upsilon_2 A_2 \varrho = \ldots$ kg/s, g/min

Bernoulli's theorem



No friction (ideal fluid)

Where the level of the head of water remains constant, the flow will be static, thus:

	point	local-	pressure-	velocity-	total-height
		heights			
n 25	0	H	0	0	=constant m, cm, mm
n 26	1	H_1	$\frac{p_1}{\varrho g}$	$\frac{\upsilon_1^2}{2g}$	
n 27	2	0	$\frac{p_2}{\varrho g}$	$\frac{\upsilon_2^2}{2g}$	

n 28 Multiplication of formulae n 25, n 26, and n 27 by g gives the energy per unit mass.

n 29 Multiplication of formulae n 25, n 26, and n 27 by G gives the total energy of mass m.

n 30 Multiplication of formulae n 25, n 26, and n 27 by ϱg gives the enrgy per unit volume.

Including friction (real fluid) – (H_w, see N 6)

n 31 $$H = H_1 + \frac{p_1}{\varrho g} + \frac{\upsilon_1^2}{2g} + H_{W_1} = \frac{p_2}{\varrho g} + \frac{\upsilon_2^2}{2g} + H_{W2}$$

HYDRAULICS

Hydrodynamics

Energy loss in circular pipes, H_w

Reynolds' number Re

This is used to ascertain whether the flow is laminar or turbulent.

n 32 $$Re = \frac{\upsilon\, d}{\nu}$$

where $Re < 2000$, the flow is laminar
where $Re > 3000$, the flow is turbulent
where $Re = 2000 \ldots 3000$, the flow is either laminar or turbulent, depending on the degree of disturbance

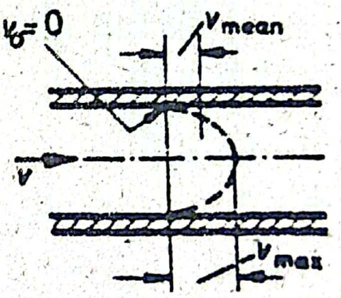
values of ν m^2/s	temp. °C	0°C	20°C	50°C	80°C	100°C
n 33	water	183×10^{-8}	100×10^{-8}	57×10^{-8}	36×10^{-8}	29×10^{-8}
n 34	lubr. oil		60×10^{-6}	15×10^{-6}		

n 35 $$H_w = \lambda \frac{l\, \upsilon^2}{2\, g\, d} = \frac{4\, f\, l\, \upsilon^2}{2\, g\, d} \qquad \text{m, cm}$$

For straight circular pipe, Re to be found from formula n 32.

Laminar flow

The loss of energy is proportional to the mean velocity υ. The velocity distribution (velocity profile) is parabolic with υ_{max} along the axis of the pipeline and $\upsilon = 0$ along its inner surface.



n 36 $$\lambda = \frac{64}{R} \quad \text{or} \quad f = \frac{16}{Re}$$

where $\lambda = 4f$, friction factors.

Turbulent flow

The velocity profile approximates to a logarithmic law.

n 37	λ for	rough calculation	0·025
n 38		more accurate calculation	$0{\cdot}02 + 0{\cdot}0018 \sqrt{\frac{1}{\upsilon\, d}}$ put in υ only in m/s d only in m

υ :	mean velocity	m/s, km/h
d :	diameter of pipeline	m, mm
l :	length of pipeline	m, mm
ν :	kinematic viscosity	m^2/s, mm^2/min

HYDRAULICS

Hydrodynamics

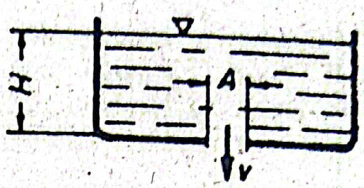
N 7

Flow of liquids from containers

Base apertures

n 39 $v = C_v \sqrt{2 g H}$

n 40 $Q = C_d A \sqrt{2 g H}$



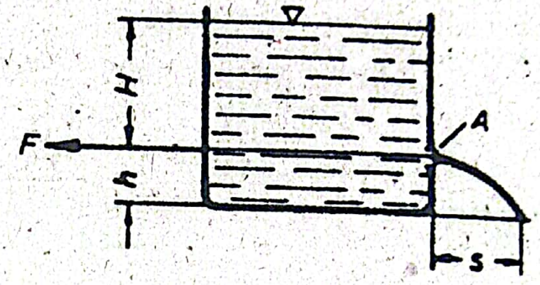
Small lateral apertures

n 41 $v = C_v \sqrt{2 g H}$

n 42 $s = 2 \sqrt{H h}$

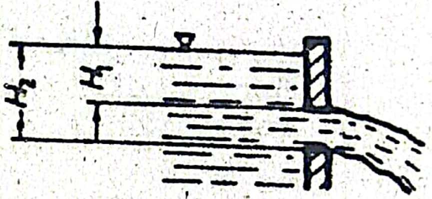
n 43 $Q = C_d A \sqrt{2 g H}$

n 44 $F = \rho Q v$



Large lateral apertures

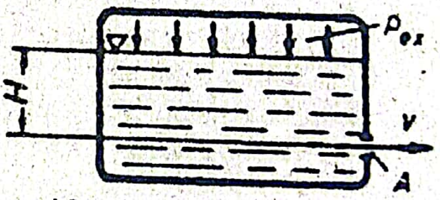
n 45 $Q = \frac{2}{3} C_d b \sqrt{2 g} \, (H_2^{3/2} - H_1^{3/2})$



Excess pressure on surface of liquid

n 46 $v = C_v \sqrt{2(g H + \frac{p_{ex}}{\rho})}$

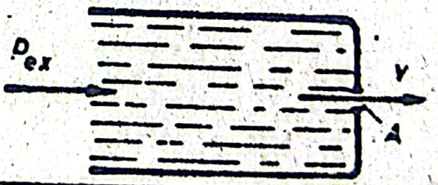
n 47 $Q = C_d A \sqrt{2(g H + \frac{p_{ex}}{\rho})}$



Excess pressure applied to an outlet point

n 48 $v = C_v \sqrt{2 \frac{p_{ex}}{\rho}}$

n 49 $Q = C_d A \sqrt{2 \frac{p_{ex}}{\rho}}$



v : outlet velocity — m/s, km/h

p_{ex} : pressure in excess of atmosph. pressure — N/m², kgf/cm²

C_d : discharge coefficient ($C_d = C_c \times C_v$)

C_c : contraction coeff. ($C_c = 0{\cdot}62$ for sharp edge apert.)
($C_c = 0{\cdot}97$ for well rounded apert.)

C_v : velocity coefficient (for water $C_v = 0{\cdot}97$)

b : width of aperture — m, cm

F : reaction force — N, kgf

Q : volume of outlet flow — m³/s, m³/h

HEAT

Heating of masses

O1

Heating of masses

O 1 **Heat (thermal energy) Q** (J, kcal)
Q represents the thermal form of energy (see M 1). 1 kilogram calorie (1 kcal) is the heat required to raise the temperature of 1 kg of water by 1 K.

O 2 **Heat per unit mass q** (J/kg, kcal/kg)
q is the quotient of quantity of heat Q and mass m:

$$q = \frac{Q}{m}$$

O 3 **Heat flow Φ** ($W = J/s$, kcal/s)
By heat flow we mean the heat transported per unit time

$$\Phi = \frac{Q}{t}$$

O 4 **Heat flow per unit area φ** ($J/(s\,m^2)$, $kcal/(h\,m^2)$)
φ is the quotient of heat Q and cross-section A it penetrates:

$$\varphi = \frac{\Phi}{A}$$

O 5 **Temperature T, t, Θ, ϑ** (K, °C)
It is one of the 3 thermal variables of state (see O 3)

O 6 **Mass m** (kg, g, t)
Explanations see M 1.

O 7 **Specific heats c and c_n respectively**
($J/(kg\,K)$, $kcal/(kg\,°C)$)
If a substance of mass m is supplied with heat Q, its temperature will increase by Δt. The specific heat of the substance may then be calculated by:

O 8

$$c = \frac{Q}{m\,\Delta t} \quad \text{resp.} \quad c_n = \frac{Q}{V_n\,\Delta t} \quad \text{and} \quad c_n = \rho_n\,c$$

For gases distinguish between c and c_n depending on whether pressure or volume is held constant during heating. Since specific heat strongly depends on temperature, a mean specific heat for the temperature range in question is used for calculations, e.g.:

O 9

$$c_{p_m}\Big|_{0\,°C}^{100\,°C}$$

The influence of pressure on c, although present, is negligibly small.

HEAT

Heat, heat flow

Calculation of heat and heat flow

o 10 heat $$Q = m\,c_{p_m}\Big|_0^t \times t$$

o 11 charge of heat $$\Delta Q = m\,c_{p_m}\Big|_0^t \times \Delta t$$

o 12 heat flow $$\Phi = m\,c_{p_m}\Big|_0^t \times t$$

symbols denote

mass	m
mean specific heat of substance	$c_{p_m}\Big\|_0^t$
temperature in °C	t

Latent heat

o 13 Latent heat may be evaluated by $l = Q/m$. The heat Q causes a change of state of mass m without an increase in temperature. The latent heat is called:

latent heat of fusion l_f, where mass m is converted from the solid to the liquid state;

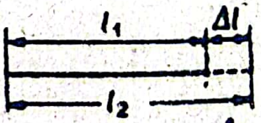
latent heat of vapourisation l_d, where mass m is converted from the boiling liquid to the dry saturated gaseous state;

heat of sublimation l_s, where mass m is converted from the solid state at pressures below triple point directly to the dry saturated gaseous state.

Expansion of solid bodies

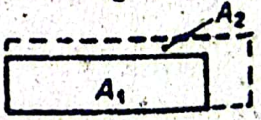
o 14 Linear expansion $l_2 = l_1[1 + \alpha(t_2 - t_1)]$

o 15 $\Delta l = l_2 - l_1 = l_1\,\alpha\,\Delta t$



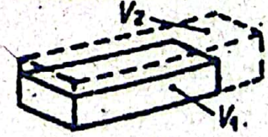
o 16 Area-expansion $A_2 = A_1[1 + 2\alpha(t_2 - t_1)]$

o 17 $\Delta A = A_2 - A_1 = A_1\,2\alpha\,\Delta t$



o 18 Volume-expansion $V_2 = V_1[1 + \gamma(t_2 - t_1)]$

o 19 $\Delta V = V_2 - V_1 = V_1\,\gamma\,\Delta t$



Expansion of liquids: volume-expansion only

Expansion of gases : see gas laws on page O 6

For explanations regarding α and γ see note on page O 9

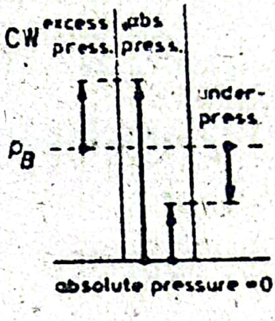
HEAT

Thermal variables of state

O 3

Thermal variables of state

For thermodynamic calculations always use absolute pressure, absolute temperature.



Pressure p (for explanations see N 1)

o 20 techn. atmosphere 1 at = 1 kgf/cm^2 = 10 m CW

o 21 physic. atmosphere 1 atm = 760 Torr

o 22 in meteorology 1 bar = 1000 mbar

(see also convers. tables on A 3 + A 5)

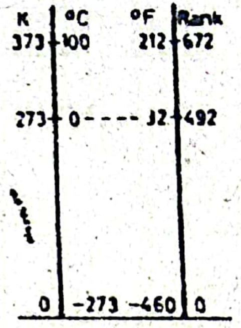
o 23 absol. pressure $p_a = p_B$ + excess press.

o 24 $= p_B$ − underpressure

o 25 (in most cases p_B = 1 at or 1 bar is accur. enough)

p_B: barometer reading

Temperature T, t, ϑ (base quantity)



scale	unit	absol. zero	melting point of ice	boil. point of water at 760 Torr
o 26 Kelvin	K, °K	0 K	273 K	373 K
o 27 Centigrade	°C	−273°C	0°C	+100°C
o 28 Fahrenheit	°F	−460°F	+32°F	+212°F
o 29 Rankine	Rank	0 Rank	492 Rank	672 Rank

Conversions:

Let t_C be the temperature in degree Centigrade
" t_F " " " " " Fahrenheit
" T_K " " " " " Kelvin
" T_R " " " " " Rankine

o 31 $t_C = \frac{5}{9}(t_F - 32)$; $t_F = \frac{9}{5} t_C + 32$; $T_K = t_C + 273$; $T_R = t_F + 460$

Volume V (units: m^3, dm^3, l)

Since the volume of a gas varies with pressure and temperature, the volume in standard condition, V_n, has been introduced. For any other volume V the corresponding pressure p and temperature T must be stated.

Volume in standard condition V_n

o 33 at standard pressure p_n = 1·013 bar = 760 Torr

o 34 and standard temperature T_n = 273 K :

o 35 $$V_n = V \frac{273\,\text{K}}{T} \times \frac{p}{1{\cdot}013\ \text{bar}}$$

Volume per unit mass v (spec. volume), (EU: m^3/kg, dm^3/g)

o 36 $$v = \frac{V}{m} = \frac{1}{\varrho}$$ (for density ϱ see N 1)

o 37 $$v_n = \frac{V_n}{m} = \frac{1}{\varrho_n}$$ (for standard density ϱ_n see Z 10)

HEAT

Gas laws

Relation between thermal variables

Gas constant R

o 38 $R = \frac{W}{m\,\Delta t}$ $\frac{\text{kJ}}{\text{kg K}}$; $\frac{\text{kgf m}}{\text{kg K}}$

Here W is the external work done by a mass m of gas, when its temperature is raised by Δt and its pressure is held constant.

General equation of state

o 39 $p = \varrho R T$

$pV = m R T$

o 40 or $p v = R T$

Conversion from one state into another:

o 41 $m R = \frac{p V}{T} = \frac{p_n V_n}{T_n} = \frac{p_1 V_1}{T_1} = \frac{p_2 V_2}{T_2}$

o 42 $V = V_n \frac{T}{T_n} \times \frac{p_n}{p}$

Thermodynamics

Thermodynamics deals with the conversion of energy (from state 1 into state 2).

	1st law	2nd law
o 43	$q_{12} = u_2 - u_1 + w_{12}$	$q_{12} = \int_{s_1}^{s_2} T\,ds$
o 44	$q_{12} = h_2 - h_1 + w_{c12}$	

The symbols denote (EU: J/kg; kcal/kg):

o 45	q_{12}		quantity of heat	per unit mass	added
o 46	$w_{12} = \int_{v_1}^{v_2} p\,dv$		external work		done
o 47	$w_{c12} = \int_{p_1}^{p_2} v\,dp$		continuous external work		done
o 48	$u = c_v\,t$	$u_2 - u_1 = c_v\,(t_2 - t_1)$	internal energy		
o 49	$h = c_p\,t$	$h_2 - h_1 = c_p\,(t_2 - t_1)$	enthalpy		
o 50	s,	$s_2 - s_1 = c_v \ln \frac{p_2}{p_1} + c_p \ln \frac{v_2}{v_1}$	entropy		

CamScanner

HEAT

Thermodynamics

Thermodynamic sign convention

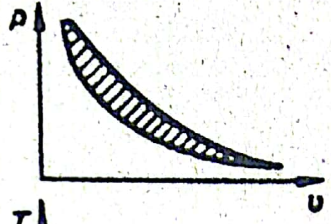
For technical applications of thermodynamics the following sign conventions apply:

quantity considered	sign
heat added	positive
heat removed	negative
mechanical work added	negative
mechanical work removed	positive
area circumvented clockwise	positive
area circumvented counter-clockwise	negative

Thermodynamic diagrams

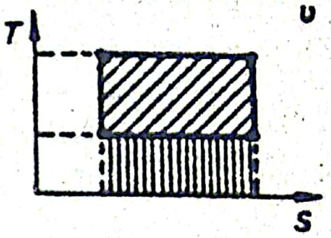
o 51 **p-v diagram (indicator diagram)**

In this diagram the mechanical work per unit mass is represented by an area.



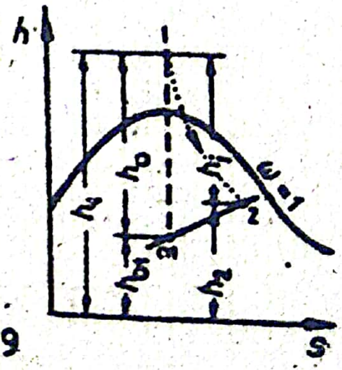
o 52 **T-s diagram**

In this diagram the heat transfer per unit mass is represented by an area.



o 53 **h-s diagram for water vapour (Mollier diagram)**

In this diagram the gained cyclic work per unit mass is represented by a distance.



o 54 isentropic heat change $h_0 = h_1 - h_{01}$

o 55 indicated heat change $h_i = h_1 - h_2$

o 56 isentropic efficiency $\eta_g = \frac{h_i}{h_0}$

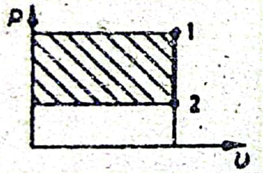
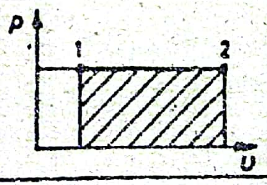
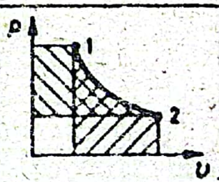
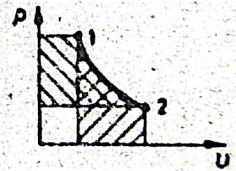
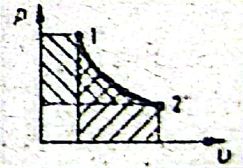
ω : see o 79

Throttling process

o 57 The following approximations apply:
$h_2 \approx h_1$, $h \approx$ constant.

o 58 For perfect gases only, since $h = c_{pm} t$, also
$t_2 \approx t_1$, $t \approx$ constant.

Variations of state: $pv = RT$

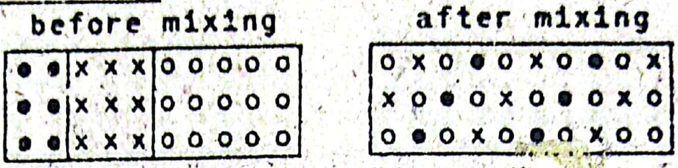
	o 63	o 62	o 61	o 60	o 59
const. quantity	v (const volume)	p (const pressure)	T (isothermal)	s (isentropic)	/ (polytropic)
equation	$p v^{\infty} =$ const.	$p v^{0} =$ const.	$p v^{1} =$ const.	$p v^{\gamma} =$ const.	$p v^{n} =$ const.
equation of state	$w_{12} = 0$ $q_{12} = u_2 - u_1$	$w_{c12} = 0$ $q_{12} = h_2 - h_1$	$u_2 - u_1 = 0$ $h_2 - h_1 = 0$ $q_{12} = w_{12} = w_{c12}$	$q_{12} = 0$ $s_2 - s_1 = 0$ $u_1 - u_2 = w_{12}$ $h_1 - h_2 = w_{c12}$	$q_{12} = u_2 - u_1 + w_{12}$ $q_{12} = h_2 - h_1 + w_{c12}$
specific heat	$c_v = c_p - R$ $q_{12} = c_{vm}(t_2 - t_1)$	$c_p = c_v \gamma$ $q_{12} = c_{pm}(t_2 - t_1)$	$c = \pm\infty$	$c = 0$ $\gamma(u_1 - u_2) = h_1 - h_2$	$c_n = c_v(\gamma - n)/(1 - n)$ $q_{12} = c_{nm}(t_2 - t_1)$
$w_{12} = \int_1^2 p\,dv$ work per unit mass	none	$p(v_2 - v_1)$ $R(t_2 - t_1)$	$p v \times \ln v_2/v_1$ $p v \times \ln p_1/p_2$	$\frac{1}{\gamma-1}(p_1 v_1 - p_2 v_2)$	$\frac{1}{n-1}(p_1 v_1 - p_2 v_2)$
$w_{c12} = \int_1^2 v\,dp$	$v(p_1 - p_2)$ $R(t_1 - t_2)$	none	$w_{c12} = w_{12}$	$\frac{\gamma}{\gamma-1}(p_1 v_1 - p_2 v_2)$ $w_{c12} = \gamma w_{12}$	$\frac{n}{n-1}(p_1 v_1 - p_2 v_2)$ $w_{c12} = n w_{12}$
areas repres. per unit mass w_{12} ▨ w_{c12} ▧					
p and T	$\frac{p_1}{p_2} = \frac{T_1}{T_2}$	–	–	$\frac{T_1}{T_2} = \left(\frac{p_1}{p_2}\right)^{\frac{\gamma-1}{\gamma}}$	$\frac{p_1}{p_2} = \left(\frac{T_1}{T_2}\right)^{\frac{n}{n-1}}$
T and v	–	$\frac{v_1}{v_2} = \frac{T_1}{T_2}$	–	$\frac{T_1}{T_2} = \left(\frac{v_2}{v_1}\right)^{\gamma-1}$	$\frac{v_2}{v_1} = \left(\frac{T_1}{T_2}\right)^{\frac{1}{n-1}}$
p and v	–	–	$\frac{p_1}{p_2} = \frac{v_2}{v_1}$	$\frac{p_1}{p_2} = \left(\frac{v_2}{v_1}\right)^{\gamma}$	$\frac{v_2}{v_1} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{n}}$
constant. =	$p_1/T_1 = p_2/T_2 = R/v$	$v_1/T_1 = v_2/T_2 = R/p$	$p_1 v_1 = p_2 v_2$	slop. steeper	polytropic.
curve	vertical line	horizont. line	rect. hyperbola	than isoth.	mostly $\gamma > n > 1$

HEAT

Gas mixtures

Gas mixture

General relations


before mixing
after mixing

o 64 $V_1 + V_2 + V_3 = V$

o 65 $m_1 + m_2 + m_3 = m$

o 66 $T, T, T \quad T$

o 67 $R_1, R_2, R_3 \quad R$

o 68 $p, p, p \quad p = p_1 + p_2 + p_3$

o 69 $\varrho_1, \varrho_2, \varrho_3 \quad \varrho$

V_1, V_2, V_3 volumes of constituents 1, 2, 3 when each exists alone at pressure p and temperature T.

p_1, p_2, p_3 pressures of constituents 1, 2, 3 when each exists alone occupies volume V at temperature T.

o 70 mass fractions $\}$ $g_i = \frac{m_i}{m}$; $\sum_1^n g_i = 1$ | gas const. of mixture $\}$ $R = \sum_1^n (g_i R_i)$

o 71 volume fractions $\}$ $r_i = \frac{V_i}{V}$; $\sum_1^n r_i = 1$ | density of mixture $\}$ $\varrho = \sum_1^n (r_i \varrho_i)$

o 72 partial pressures $\}$ $p_i = r_i p$ | total pressure $\}$ $p = \sum_1^n p_i$

Interrelation between mass- and volume fractions

o 73 $g_i = r_i \frac{\varrho_i}{\varrho} = \eta \frac{R}{R_i}$; $r_i = g_i \frac{\varrho}{\varrho_i} = g_i \frac{V_i}{V} = g_i \frac{R_i}{R}$

Specific heat of mixture

o 74 $c = \sum_1^n (g_i c_i) \quad \frac{J}{kg\,K}$; $\frac{kcal}{kg\,^oC}$ | $c_n = \sum_1^n (r_i c_{ni}) \quad \frac{J}{m^3 K}$; $\frac{kcal}{m^3\,^oC}$

Mixture of air and water vapour (humid air)

o 75 total pressure of mixt. (accord. to o 72) $p = p_a + p_v$

o 76 air satur. with water vapour ($\varphi = 1 \hat{=} 100\%$) $p_v = p_{vs}(t)$

o 77 unsaturated air / superheated vapour $\}$ $(\varphi < 1) \quad p_v < p_{vs}(t)$

o 78 relative humidity $\}$ $\varphi = \frac{p_v}{p_{vs}(t)} = 100 \frac{p_v}{p_{vs}(t)} \% = \frac{\omega}{\omega_s \ (t)}$

o 79 spec. humidity of mixt. (moist. const.) $\}$ $\omega = \frac{\text{mass of water vapour } m_v}{\text{mass of dry air } m_a} \quad \frac{kg}{kg}; \frac{g}{kg}$

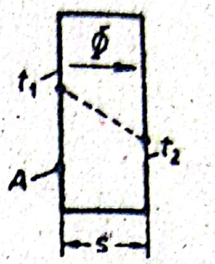
o 80 total pressure $p = p_a + \varphi\, p_{vs}(t)$

p_v	partial pressure of	vapour	p_{vs} : saturation pressure at temperature t of mixture
p_a		air	

HEAT

Heat transmission

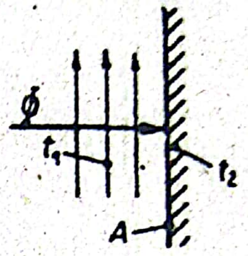
Heat conduction



For a body with a temperature gradient, the heat flow from the high to the low temperature region is:

o 81 $$\Phi = \frac{\lambda}{s} A \, (t_1 - t_2) \qquad W, \frac{J}{s}$$

Heat exchange

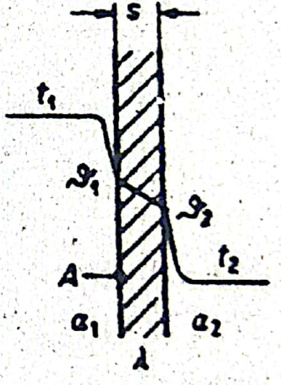


Where a temperature difference exists between a streaming fluid and the surface of the pipeline or container, the heat flow through the interface is:

o 82 $$\Phi = \alpha \, A \, (t_1 - t_2) \qquad W, \frac{J}{s}$$

Heat transfer

Where the temperatures of two substances separated by a partition wall differ, the heat flow from the warmer to the cooler substance is:



o 83 $$\Phi = k \, A \, (t_1 - t_2) \qquad W, \frac{J}{s}$$

The overall heat transfer u, is determined by

o 84 $$k = \frac{1}{\frac{1}{\alpha_1} + \frac{s}{\lambda} + \frac{1}{\alpha_2}}$$. Here symb. denote

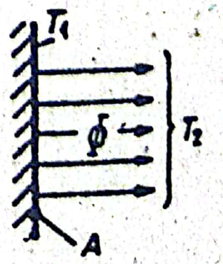
$\left.\begin{matrix}\alpha_1\\ \alpha_2\end{matrix}\right|$ heat transfer coefficient at $\left|\begin{matrix}\text{left}\\ \text{right}\end{matrix}\right|$ of partition

λ thermal conductivity of partition substance. Thus, the heat flow may also be written as:

o 85 $$\Phi = \alpha_1 A \, (t_1 - \vartheta_1) = \frac{\lambda}{s} A \, (\vartheta_1 - \vartheta_2) = \alpha_2 \, A \, (\vartheta_2 - t_2)$$

A layer of dirt on either side of the partition wall will reduce the heat transfer coefficient and cause an additional drop of temperature.

Heat radiation



The heat flow emitted by a body of surface A and absolute temperature T_1 into a space of absolute temperature T_2 is:

o 86 $$\Phi = C \, A \, (T_1^4 - T_2^4) \qquad W, \frac{J}{s}$$

explanation of symbols on page O 9

HEAT

Heat transmission

Heat exchanger

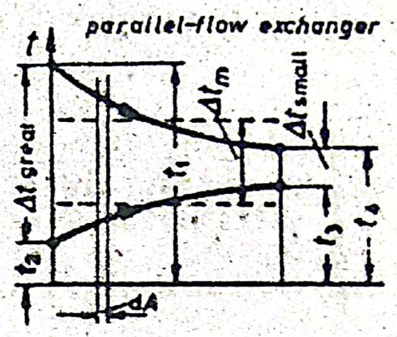
Heat exchanger

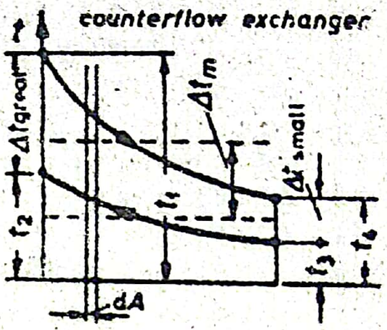
In a heat exchanger in operation one substance is cooled down and the other one warmed up, when travelling along the exchanger heat-transfer area.

For both a parallel-flow and a counterflow exchanger the heat flow may be calculated by:

87 $$\Phi = k\, A\, \Delta t_m \qquad W,\ \frac{J}{s}$$

(for determinat. of u see O 84)


parallel-flow exchanger


counterflow exchanger

Logarithmic mean temperature difference

This value is often used, since it is obtained theoretically on the assumption of constant u.

88 $$\Delta t_{m_{log}} = \frac{\Delta t_{great} - \Delta t_{small}}{\ln \frac{\Delta t_{great}}{\Delta t_{small}}} \qquad K$$

Arithmetic mean Δt_m
(approximate value)

89 $$\Delta t_m \approx \frac{t_1 + t_2}{2} - \frac{t_4 + t_3}{2} \qquad K$$

Symbols used (including page O 8)

		for values see
λ : thermal conductivity	W/(m K)	Z 5...Z 10
α : heat transfer coefficient (film coefficient)	W/(m² K)	Z 13
k : overall transfer heat coeff.	W/(m² K)	
C : radiation coefficient	kW/(m² K⁴)	Z 13
		Z 13

90 $$C = \varepsilon\, \sigma = \frac{\text{energy emitted by surface}}{\text{energy emitted by black body}} \times \sigma$$

C : Stefan-Boltzmann constant – 56·7 × 10^{-12} kW/(m² K⁴)

$t_1 - t_2$: temperature difference

Symbols used on page O 2

α : linear
2α : area
$\gamma = 3\alpha$: volume
} coefficient of expansion $\frac{1}{K}, \frac{1}{^\circ C}$

(for values see Z 12)

STRENGTH

General terms

P 1

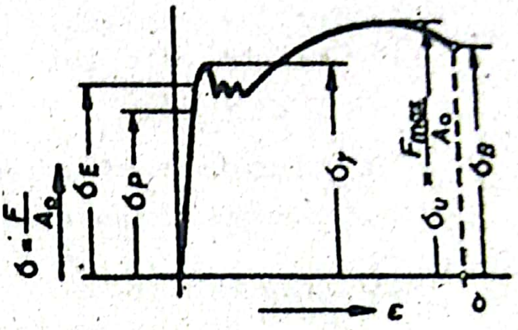
General

Stress

Stress is the ratio of applied force (tensile or compressive force) and cross section.

$$\sigma = \frac{F}{A} \qquad N/m^2 \; ; \; N/mm^2 \; ; \; kgf/mm^2$$

Stress-strain diagram for mild steel



σ_P: proportional limit
σ_E: elastic limit
σ_Y: yield point
- σ_{Yt} yield point under tensile stress
- σ_{Yc} yield point under compr. stress

σ_u: ultimate stress (tensile strength)
σ_B: breaking stress
δ : elongation
ε : strain
A_0 : original cross-sectional area (at zero strain)

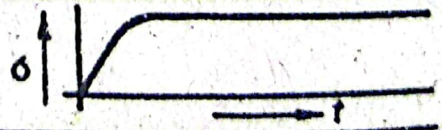
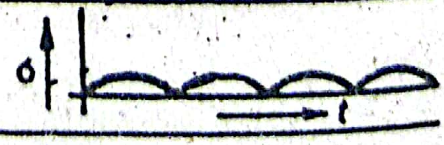
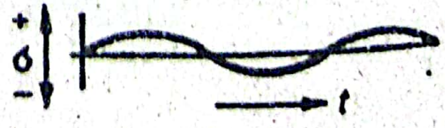
Permissible stress (allowable stress)

Must be below the elastic limit σ_E, thus the permissible stress is:

$$p_t = \frac{\sigma_Y}{S}$$

σ_Y : yield strength of material
S : safety factor, always greater than 1. Its extent is governed by the type of load involved.

Loads

type	nature of stress	load diagram
No: I	dead	
No: II	undulating	
No: III	alternating	

STRENGTH

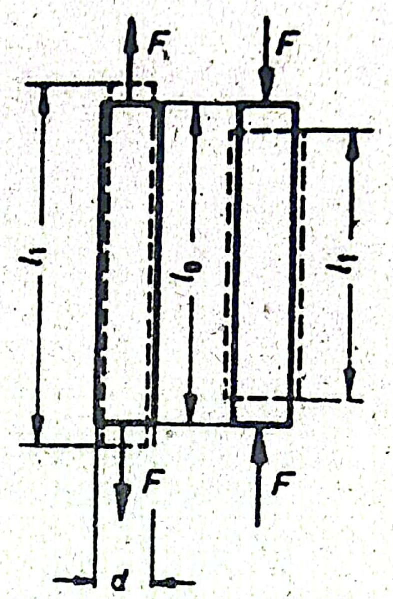
Tension, Compression

Tension and compression

Tensile or compressive stress

p 2 $\sigma_t = \frac{F}{A} \leqq p_t$

p 3 $\sigma_c = \frac{F}{A} \leqq p_c$



Modulus of elasticity E

p 4 $E = \frac{\sigma}{\varepsilon}$

Coefficient of elongation α

p 5 $\alpha = \frac{1}{E} = \frac{\varepsilon}{\sigma}$

Increase in length Δl

p 6 $\Delta l = l_1 - l_0 = l_0 \alpha \sigma$

Strain ε

p 7 $\varepsilon = \frac{\Delta l}{l_0} = \alpha \sigma$

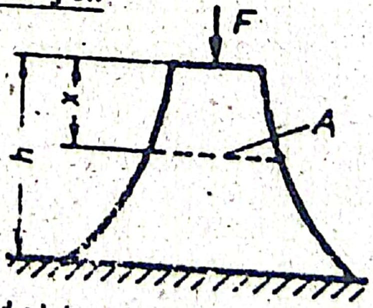
Elongation δ

p 8 $\delta = \frac{\Delta l \times 100}{l_0} \% = \frac{\sigma}{E} 100 \% = \alpha \sigma 100 \%$

(δ_5 for $l_0 = 5d$, δ_{10} for $l_0 = 10d$)

Columns of uniform compressive strength+)

At any cross section distant x from the top, the required area A may be calculated as follows:



p 9 $A = \frac{F}{p_c} e^{\frac{\rho g}{p_c} x}$

+) taking into account dead weight of column

---

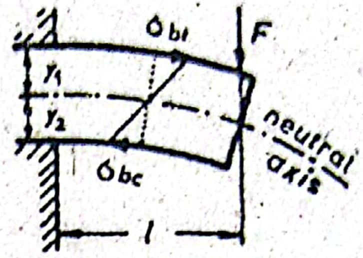
p_t: permissible tensile stress (for values see Z 14)
p_c: perm. compressive stress (for values see Z 14)
e : 2·718 281 ...

STRENGTH

Bending

Bending

(moment of resistance)



Section modulus z

p 10 $z = \frac{I}{y_{max}}$

Bending stress σ_{bt}

p 11 $\sigma_{bt} = \frac{M\, y_{max}}{I} \leq p_{bt}$

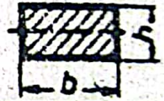
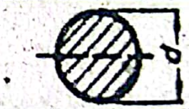
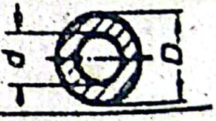
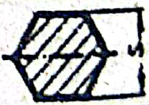
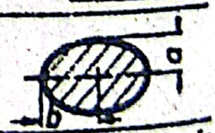
If $y = y_1 = y_2$ (neutral axis is also axis of symmetry):

p 12 $\sigma_{bt} = \frac{M}{z}$

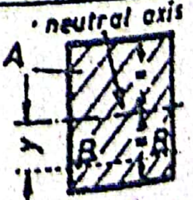
Max. bending moment M (for cantilever)

p 13 $M = F\,l$

Second moments of area, section modulus and maximum bending stress

	second moment of area I	section modulus z	max. bending stress σ_{bc}, σ_{bt}	cross section A
p 14	$\frac{b\,h^3}{12}$	$\frac{b\,h^2}{6}$	$\frac{6\,M}{b\,h^2}$	
p 15	$\frac{\pi\,d^4}{64}$	$\frac{\pi\,d^3}{32} \approx \frac{d^3}{10}$	$\approx \frac{10\,M}{d^3}$	
p 16	$\frac{\pi}{64}(D^4 - d^4)$	$\frac{\pi}{32} \times \frac{D^4 - d^4}{D}$	$\approx \frac{10\,M\,D}{D^4 - d^4}$	
p 17	$\frac{5\sqrt{3}\,s^4}{144}$	$\frac{5\sqrt{3}\,s^3}{72}$	$\frac{24\sqrt{3}\,M}{5\,s^3}$	
p 18	$\frac{\pi\,a^3\,b}{4}$	$\frac{\pi\,a^2\,b}{4}$	$\frac{4\,M}{\pi\,a^2\,b}$	

Parallel axis theorem



p 19 second moment of area $I_{BB} = I + A\,y^2$

I : {second moment of} —·— (see J 10)

I_{BB}: {area about axis} ----- (see J 10)

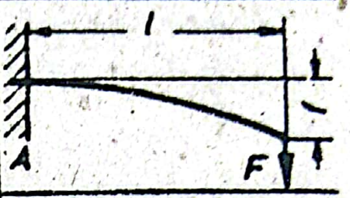
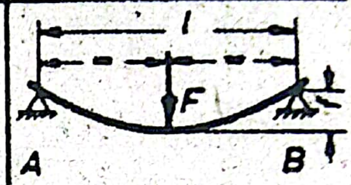
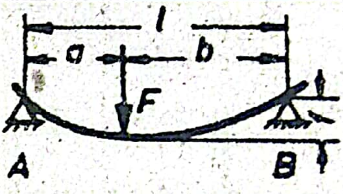
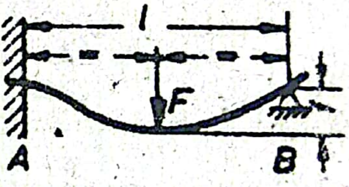
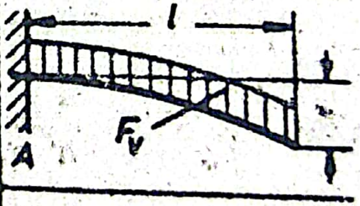
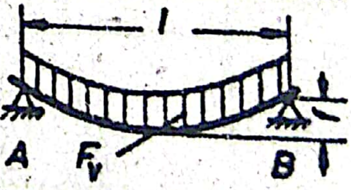
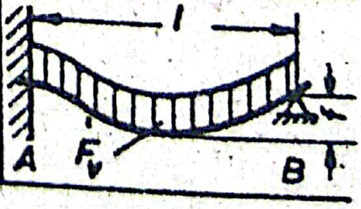
y_{max} : max. distance between neutral axis and surface [fibre

STRENGTH

Beams

P 4

Bending of uniform beams

	support reactions reaction force at A	 at B	 reaction moment at A	max. bendg. moment M_b	deflection f	loading
p 20	F	–	$F\,l$	$F\,l$	$\frac{F\,l^3}{3\,E\,I}$	
p 21	$\frac{F}{2}$	$\frac{F}{2}$	–	$\frac{1}{4}\,F\,l$	$\frac{F\,l^3}{48\,E\,I}$	
p 22	$\frac{F\,b}{l}$	$\frac{F\,a}{l}$	–	$\frac{F\,a\,b}{l}$	$\frac{F\,a^2\,b^2}{3\,E\,I\,l}$	
p 23	$\frac{11}{16}\,F$	$\frac{5}{16}\,F$	$\frac{3}{16}\,F\,l$	$\frac{3}{16}\,F\,l$	$\frac{7\,F\,l^3}{768\,E\,I}$	
p 24	F_v	–	$\frac{1}{2}\,F_v\,l$	$\frac{1}{2}\,F_v\,l$	$\frac{F_v\,l^3}{8\,E\,I}$	
p 25	$\frac{F_v}{2}$	$\frac{F_v}{2}$	–	$\frac{1}{8}\,F_v\,l$	$\frac{5\,F_v\,l^3}{384\,E\,I}$	
p 26	$\frac{5}{8}\,F_v$	$\frac{3}{8}\,F_v$	$\frac{1}{8}\,F_v\,l$	$\frac{1}{8}\,F_v\,l$	$\frac{F_v\,l^3}{185\,E\,I}$	

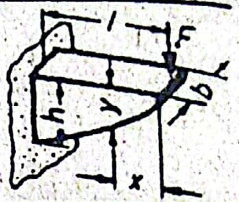
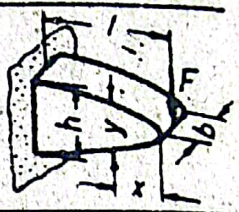
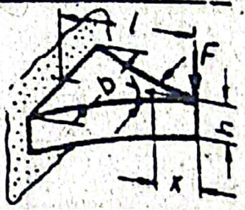
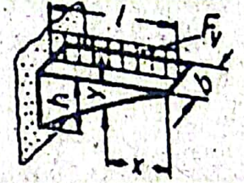
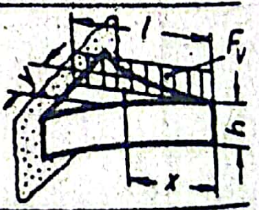
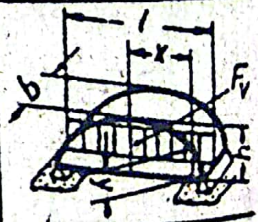
F : concentrated load (N, kN, kgf)
$F_v = w \times l$ uniformly distributed load (N, kN, kgf)

CamScanner

STRENGTH

Beams

Beam of uniform strength

	maximum section dimension	typical dimension y	maximum deflection f	beam type
p 27	$h = \sqrt{\frac{6\,F\,l}{b\,p_{bt}}}$	$\sqrt{\frac{6\,F\,x}{b\,p_{bt}}}$	$\frac{8\,F}{b\,E}\left(\frac{l}{h}\right)^3$	 
p 28	$b = \frac{6\,F\,l}{h^2\,p_{bt}}$	$\frac{6\,F\,x}{h^2\,p_{bt}}$	$\frac{6\,F}{b\,E}\left(\frac{l}{h}\right)^3$	
p 29	$h = \sqrt{\frac{3\,F_v\,l}{b\,p_{bt}}}$	$x\sqrt{\frac{3\,F_v}{b\,l\,p_{bt}}}$	$\frac{3\,F_v}{b\,E}\left(\frac{l}{h}\right)^3$	
p 30	$b = \frac{3\,F_v\,l}{h^2\,p_{bt}}$	$\frac{3\,F_v\,x^2}{h^2\,l\,p_{bt}}$		
p 31	$h = \sqrt{\frac{3\,F_v\,l}{4\,p_{bt}}}$	$\sqrt{\frac{3\,F_v\,l}{4\,b\,p_{bt}}\left(1-\frac{4x^2}{l^2}\right)}$	$\frac{F_v\,l^3}{64\,E\,I}$	

F : concentrated load N, kN, kgf

$F_v = w \times l$ uniformly distributed load N, kN, kgf

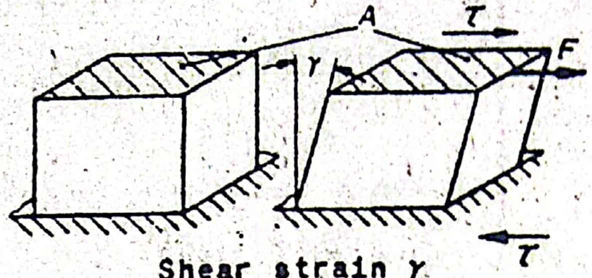
p_{bt} : permissible bending stress (see Z 14)

STRENGTH

Shear

P 6

Shear



Shear stress τ

p 32 $\tau = \frac{F}{A} \leqq p_q$

Shear modulus G

p 33 $G = \frac{\tau}{\gamma} = 0 \cdot 385\, E$

Shear coefficient β

p 34 $\beta = \frac{1}{G} = \frac{\gamma}{\tau}$

Shear strain γ

$\gamma = \beta \tau$

Ultimate shear stress τ_u

p 35 $\tau_u = \frac{F_{max}}{A} \approx 0 \cdot 8\, \sigma_u$

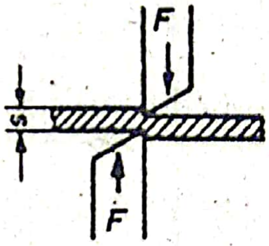
Shear strength p_q

type of load (see P 1)		$p_q \approx$	
	dead		$\sigma_y / 1 \cdot 5$
	undulating		$\sigma_y / 2 \cdot 2$
	alternating		$\sigma_y / 3 \cdot 0$

Shearing force F

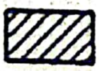
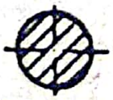
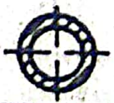
applied by	
guillotine shears	cutting tools (punch...)
$F \approx 1 \cdot 7\, \tau_u\, l\, s$	$F \approx 1 \cdot 7\, \tau_u\, l_u\, s$

p 36



Practical application of shearing stress

Shear always occurs with bending. Formula p 32 therefore includes a coefficient depending on the cross section. (Only with very short beams may the bending moment be ignored.)

cross section A			
p 37 shear stress $\tau =$	$\frac{3}{2} \frac{F}{A}$	$\frac{4}{3} \frac{F}{A}$	$2 \frac{F}{A}$

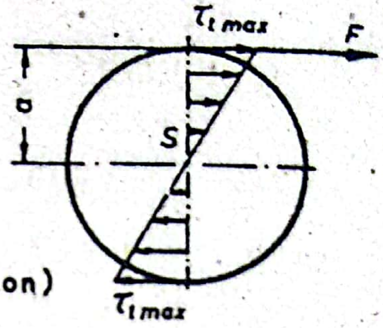
p_q, p_{ql}: allowable shear stress (for values see Z 14)
F : shearing force
l : length of cut
F_{max} : breaking load
l_u : perimeter of cut

STRENGTH

Torsion

Torsion

Shear stress due to torsion τ_t



p 38 $\tau_t = \frac{M}{W_t} \leqq p_{qt}$

Torque M

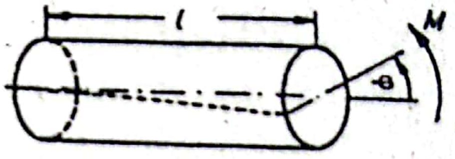
p 39 $M = \frac{P}{\omega} = \frac{P}{2\pi n} = F\,a$

Torsion stress constant W_t (for circular section)

p 40 $W_t = \frac{J}{a}$

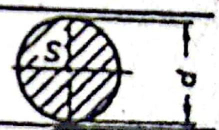
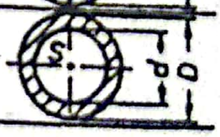
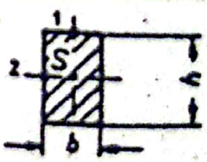
a : distance between extreme surface fibre and centre of gravity S

Angle of twist φ



p 41 $\varphi = \frac{M\,l}{J\,G}$

Polar 'moment of inertia', torsion stress constant and maximum shear stress

	moment of inertia J	torsion stress const. W_t	max. shear stress τ_t	cross section A
p 42	$\frac{\pi d^4}{32}$	$\frac{\pi d^3}{16}$	$\approx 5{\cdot}1 \frac{M}{d^3}$	
p 43	$\frac{\pi}{32}(D^4 - d^4)$	$\frac{\pi}{16}\,\frac{D^4 - d^4}{D}$	$\approx 5{\cdot}1\,\frac{M}{D^3}\,\frac{1}{1-\delta^4}$	
p 44	–	in 1: $\frac{2}{9}\,b\,h^2$	$\frac{M(3h + 1{\cdot}8b)}{b^2h^2}$	
p 45		in 2: $\frac{2}{9}\,b^2\,h$	$\frac{M(3b + 1{\cdot}8h)}{b^2h^2}$	

δ-table of annulus cross sections (see p 43)

p 46						
$\delta = \frac{d}{D}$	0·5	0·6	0·7	0·8	0·9	0·95
$\frac{1}{1-\delta^4}$	1·0667	1·1489	1·3160	1·6938	2·9078	5·3910

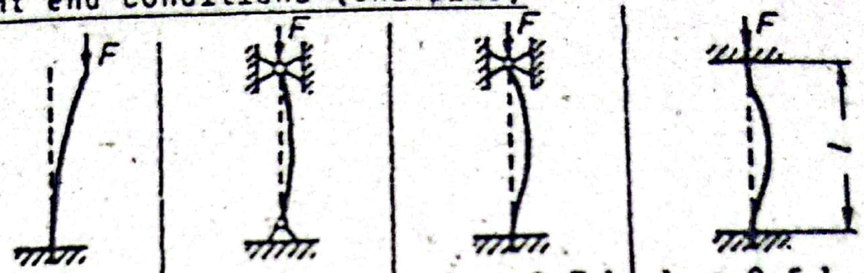
p_{qt}: perm. shear stress due to torsion (for values see Z14)

STRENGTH

Buckling

P 8

Different end conditions (examples)



p 47 $l_0 = 2\,l$ | $l_0 = l$ | $l_0 = 0{\cdot}7\,l$ | $l_0 = 0{\cdot}5\,l$

Preliminary dimensioning

Calculate second moment of area, initially assuming the Euler formula to be applicable $I = \frac{F\,l_0^2}{\pi^2 E}$

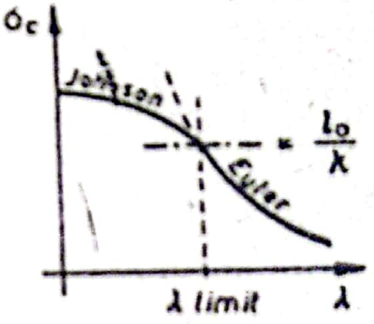
p 48 Determine cross-sectional shape and calculate preliminary dimensions according to P 3.

Final dimensioning

p 49 Slenderness $\lambda = \frac{l_0}{k} = l_0\sqrt{\frac{A}{I}}$

If $\lambda < \pi\sqrt{\frac{2E}{p_c}}$, use Johnson formula

If $\lambda > \pi\sqrt{\frac{2E}{p_c}}$, use Euler formula



p 50 **Johnson formula:** $\sigma_c = p_c - \frac{p_c^2}{4\pi^2 E}\lambda^2 = a - b\lambda^2$

Approximate values:

material	MN/m² a	MN/m² b	λ limit
mild steel	276	$9{\cdot}2 \times 10^{-3}$	122
cast iron	414	$43{\cdot}1 \times 10^{-3}$	70
timber	28	$0{\cdot}9 \times 10^{-3}$	140

p 51 **Euler formula:** $\sigma_c = \frac{\pi^2 E I}{l_0^2 A}$

If $\sigma_c < \frac{F}{A}$, repeat final dimensioning using larger cross section.

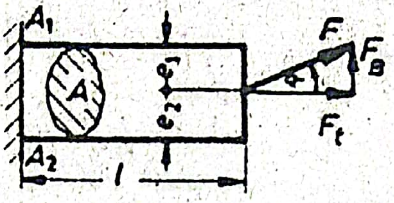
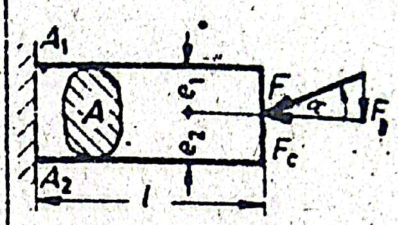
p_c : permissible compressive stress (for values see Z 14)
σ_c : actual buckling stress
F : actual load
k : section radius of gyration

STRENGTH

Composite stresses

Combination of normal stresses

According to the law of superposition, stresses due to tension and bending or compression and bending should be added algebraically using the plus sign for tension and the minus sign for compressive stresses:

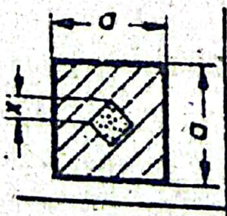
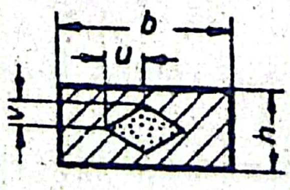
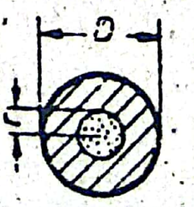
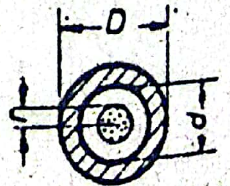
	resultant stress σ_{res}	
point	tension and bending	compress. + bending
		
p 52 A_1	$+\frac{F_t}{A} - \frac{F_B\, l\, e_1}{I} \leq p_t$	$-\frac{F_c}{A} + \frac{F_B\, l\, e_1}{I} \leq p_c$
p 53 A_2	$+\frac{F_t}{A} + \frac{F_B\, l\, e_2}{I} \leq p_t$	$-\frac{F_c}{A} - \frac{F_B\, l\, e_2}{I} \leq p_c$

p 54 $F_t = F \cos \alpha$ $F_B = F \sin \alpha$ $F_c = F \cos \alpha$

By displacing the neutral fibre i.e. by altering the cross section shape, equal tension and compressive stresses can be obtained at A_1 and A_2 (i.e. $e_1 \neq e_2$). Check long girders, under compression, with unfavourable ratio λ, for buckling (p 49, p 50).

Limit cross section for similar types of stress

Where a tension (or compression) force is applied at a point within the dotted core area, only tension (or compression) forces will occur over the whole cross section. If applied at any other point, bending stress, i.e. simultaneous tension and compression stress will occur.

			
p 55 $x = \frac{a}{6}$	$u = \frac{b}{6}\,;\; v = \frac{h}{6}$	$r = \frac{d}{8}$	$r = \frac{D}{8}\left[1+\left(\frac{d}{D}\right)^2\right]$

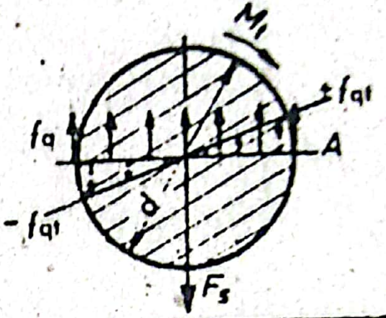
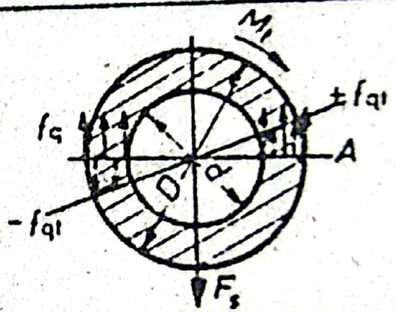
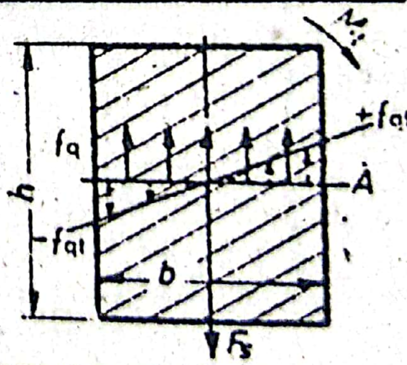
STRENGTH

Composite stresses

P 10

Combination of shear stresses

According to the law of superposition, shear and torsional shear stresses should be added algebraically:

	maximum shear stress τ_{res} at point A	cross section
p 56	$\frac{5.1\,M_t}{d^3} + \frac{1.7\,F_s}{d^2} \leqq p_{qt}$	 M_t $\pm f_{qt}$ f_q A $-f_{qt}$ d F_s
p 57	$\frac{5.1\,M_t\,D}{D^4 - d^4} + \frac{2.55\,F_s}{D^2 - d^2} \leqq p_{qt}$	 M_t $\pm f_{qt}$ f_q A $-f_{qt}$ D d F_s
p 58	$\frac{4.5\,M_t}{b^2\,h} + \frac{1.5\,F_s}{b\,h} \leqq p_{qt}$	 M_t $\pm f_{qt}$ f_q A h $-f_{qt}$ b F_s

p_{qt} : permissible shear stress

f_q : shear stress

f_{qt} : torsional shear stress

M_t : applied torque

(see Z 14)

Combination of bending and shear stresses

Where bending and shear stresses occur simultaneously,

the equivalent stress σ_v and
the equivalent moment M_v

are used for calculations.

The equivalent stress is the bending stress producing the same effect, as the two other stresses (i.e. bending stress and shear stress) combined. The equivalent moment has been defined accordingly.

Circular shafts

p 59 equivalent stress $\sigma_v \leqq p_{bt}$

p 60 $$\sigma_v = \sqrt{f_{bt}^2 + 3(\alpha_0 f_{qt})^2}$$

p 61 equivalent moment $$M_v = \sqrt{M_b^2 + 0{\cdot}75(\alpha_0 M)^2}$$

For dimensioning the shaft, first determine the section modulus by

p 62 $$z_b = \frac{M_v}{p_{bt}}$$

Then, the diameters may be calculated according to P 3.

---

f_{bt} : tensile stress due to bending
f_{qt} : shear stress due to torsion
M_b : bending moment
M : torque
α_0 : stress ratio (below the fatique criteria),
  $\alpha_0 \approx 1$, where order of stress is equal for torsion and bending
  $\alpha_0 \approx 0{\cdot}7$, where torsion is dead or undulating and bending is alternating (see P 1)

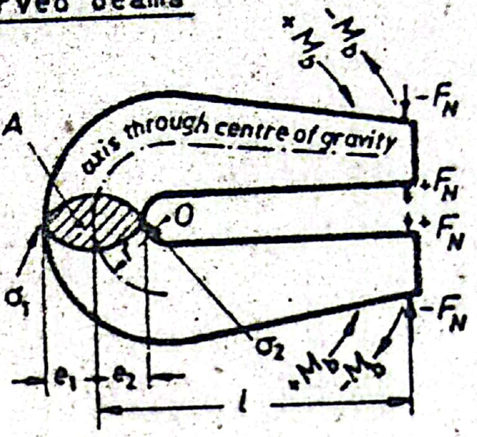
CamScanner

STRENGTH

Curved beams

Stresses in curved beams

In a curved beam the greatest stress due to the normal force F_N and bending moment M_b appears on cross section A. The edge stresses are as follows (depending on the direction of loading, M_b and F_N will be either positive or negative):

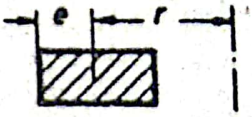


p 63 $$\sigma_1 = \frac{F_N}{A} + \frac{M_b}{A\,r} + \frac{M_b}{x\,A\,r} \times \frac{e_1}{r + e_1} \leqq p_t$$

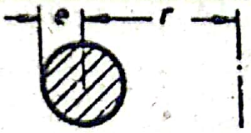
p 64 $$\sigma_2 = \frac{F_N}{A} + \frac{M_b}{A\,r} - \frac{M_b}{x\,A\,r} \times \frac{e_2}{r - e_2} \leqq p_t$$

Formula for x

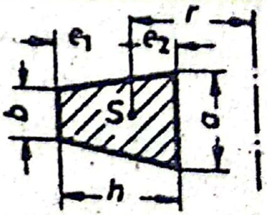
p 65 $$x = \frac{1}{3}\left(\frac{e}{r}\right)^2 + \frac{1}{5}\left(\frac{e}{r}\right)^4 + \frac{1}{7}\left(\frac{e}{r}\right)^6 + \dots$$



p 66 $$x = \frac{1}{4}\left(\frac{e}{r}\right)^2 + \frac{1}{8}\left(\frac{e}{r}\right)^4 + \frac{5}{64}\left(\frac{e}{r}\right)^6 + \dots$$



p 67 $$x = \frac{2\,r}{(a+b)\,h}\left\{\left[b + \frac{a-b}{h}\,(r+e_1)\right] \times \ln\frac{r+e_1}{r-e_2} - (a-b)\right\} - 1$$



For position of centroid S, see K 7

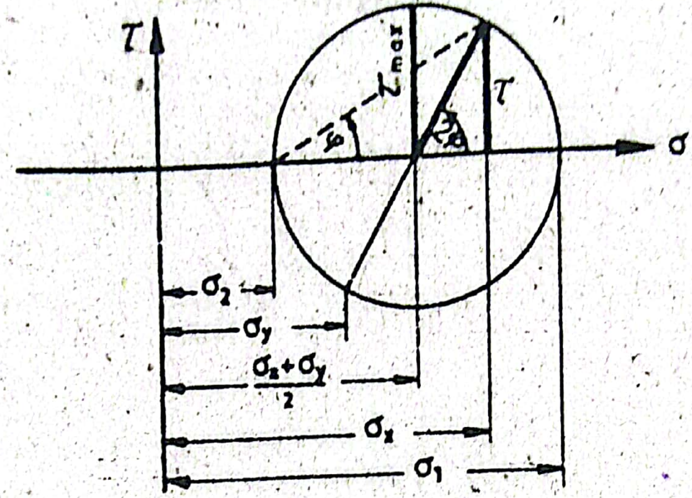
Calculated values of x

	$\frac{e}{r}$	0·1	0·3	0·5	0·7	0·8	0·9	0·95
p 68	x ▨	0·0033	0·0317	0·0986	0·239	0·373	0·636	0·928
p 69	x ⊘	0·0025	0·0236	0·0718	0·167	0·250	0·393	0·524

p 70 $M_b = F_N\,l$;

p_t : permissible tension see Z 14

Mohr's stress circle



Principal stresses

p 71 $$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2}$$

Maximum shear stresses

p 72 $$\tau_{max} = \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2}$$

Direction of principal stresses
(angles of principal planes)

p 73 $$\tan 2\varphi = \frac{2 \times \tau}{\sigma_x - \sigma_y} \quad x)$$

always applicable: $2\varphi < 90^\circ$

Direction of maximum shear stress

p 74 $$\cot 2\varphi = \frac{2 \times \tau}{\sigma_y - \sigma_x}$$

Shearing stresses will be zero on principal planes.

Maximum shear stresses occur at angles of 45° to the planes of principal stresses.

x) This solution will give two angles, this means that the principal stresses and the maximum shear stresses will occur at the point where two lines intersect at right angles.

MACHINE PARTS

Spur gears

Q1

Spur gears employing involute-tooth system

Tooth calculation for strength

q 1	pitch	$t\ (\mathrm{mm}) = 97{\cdot}6\sqrt[3]{M \times \frac{1}{\beta \times z \times p_{bt}}}$
q 2		$t\ (\mathrm{mm}) = 877\sqrt[3]{\frac{P}{n} \times \frac{1}{\beta \times z \times p_{bt}}}$

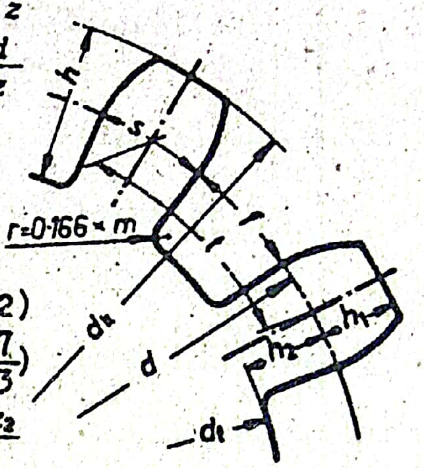
M in kgf m, P in H.P., n in revs/min, p_{bt} in MN/m^2

q 3	width of tooth	$b = \beta\, t$

Tooth dimensions (mm)

q 4	pitch	$t = m\pi = \frac{\pi d}{z}$
q 5	module	$m = \frac{t}{\pi} = \frac{d}{z}$
q 6	pitch circle diameter	$d = m\, z$
q 7	depth of teeth	$h = 2{\cdot}166\, m$
q 8	addendum	$h_1 = m$
q 9	dedendum	$h_2 = 1{\cdot}166\, m$
q 10	outside circle diameter	$d_k = m(z + 2)$
q 11	root circle diam.	$d_f = m(z - \frac{7}{3})$
q 12	distance between centres	$a = m\,\frac{z_1 + z_2}{2}$
q 13	tooth spacing	$f = 0{\cdot}5125\, t$

root width of tooth
$s = 0{\cdot}4875\, t$



Gear wheel dimensions (mm)

q 14	bore	$d = \sqrt[3]{\frac{5{\cdot}1\, M}{p_{qt}}}$
q 15	hub diameter	$D = 1{\cdot}8\, d + 20$
q 16	hub length	$l \geqq 1{\cdot}5\, d$
q 17	number of spokes	$i = 0{\cdot}166\sqrt{m\, z}$
q 18	tooth wheel rim	$x = 0{\cdot}5\, t$

put in:
M in kgf mm
p_{qt} in kgf/mm^2
d in mm
m in mm

---

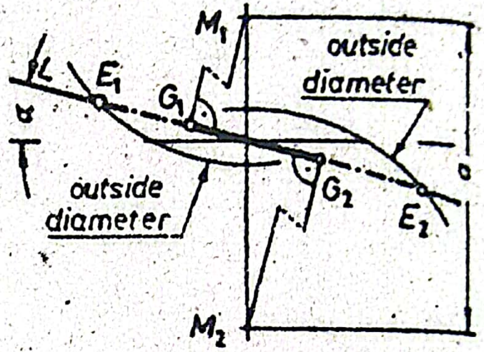
z : number of teeth | M : torque (see P 7)
p_{qt}: permissible torsion stress (for values see Z 14)
β : rough cut teeth 2...2·5; for machined teeth 2·5...3
for ground teeth 3...5

Spur gears with involute teeth

	pressure angle α / minimum number of teeth	made with undercutting 15°	made with undercutting 20°	made without undercutting 15°	made without undercutting 20°
q 19	on one wheel	14	11	20	13
q 20	on both wheels	38	27	50	30

Interference

Where the points of intersection E_1 and E_2 on the line L fall outside the line $\overline{G_1G_2}$ interference will occur, this can be eliminated by undercutting the teeth.



Arc of contact ε

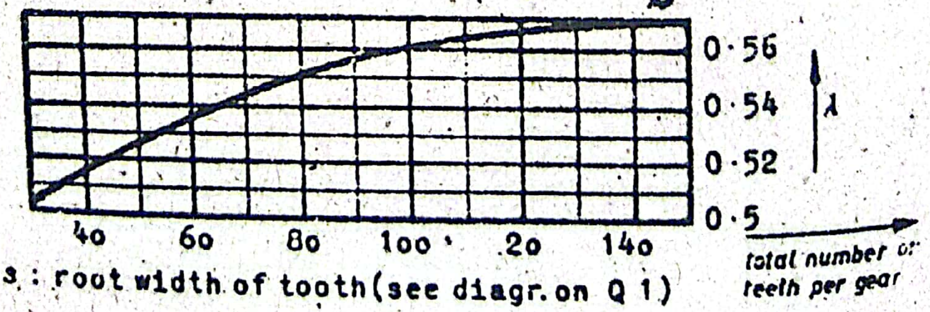
q 21 $$\varepsilon = \frac{\text{area of contact}}{\text{reduced pitch}} = \frac{\overline{E_1 E_2}}{t \times \sin(90^\circ - \alpha)} > 1$$

Undeecutting (Foelmer formula)

Can be manufactured with standard tools for $\alpha = 15^\circ$

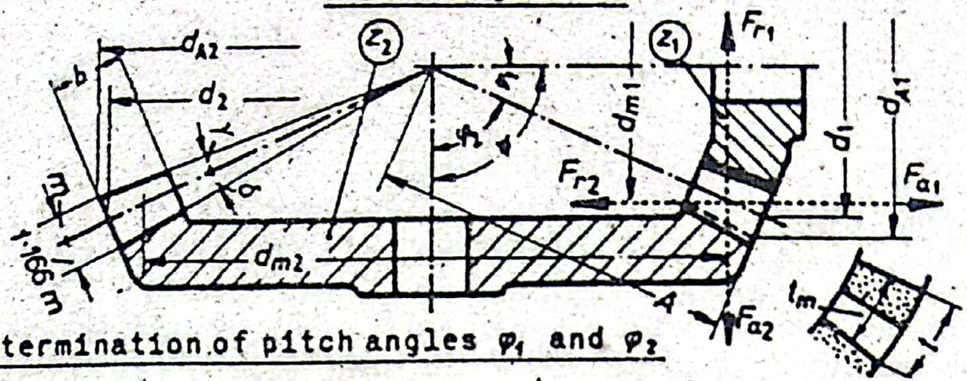
Limit number of teeth: $z_1 \gtrless 8$ and $z_2 \gtrless 26$

		dimensions small wheel z_1	dimensions large wheel z_2
q 22	outside circle diam.	$d_{k1} = (z_1 + 3)\ m$	$d_{k2} = (z_2 + 1{\cdot}8)\ m$
q 23	root circle diam.	$d_{f1} = (z_1 - 1{\cdot}134)\ m$	$d_{f2} = (z_2 - 2{\cdot}334)\ m$
q 24	root width of tooth	$s_1 = 1{\cdot}89\ m$	$s_2 = 1{\cdot}57\ m$
q 25	axial spacing	$a = \left(\frac{z_1 + z_2}{2} + \lambda\right) m$	



s : root width of tooth (see diagr. on Q 1)

Bevel gears



Determination of pitch angles φ_1 and φ_2

		bevel gear z_1	bevel gear z_2
q 26	$\psi \gtrless 90^\circ$	$\cot\varphi_1 = \dfrac{\frac{z_2}{z_1} + \cos\psi}{\sin\psi}$	$\cot\varphi_2 = \dfrac{\frac{z_1}{z_2} + \cos\psi}{\sin\psi}$
q 27	$\psi = 90^\circ$	$\cot\varphi_1 = \dfrac{z_2}{z_1}$	$\cot\varphi_2 = \dfrac{z_1}{z_2}$

Tooth dimensions

q 28 tooth crest angle from $\tan\gamma = \dfrac{m}{A}$

q 29 tooth root angle from $\tan\delta = \dfrac{1{\cdot}166\,m}{A}$

q 30 distance $A = \dfrac{z_1\,m}{2\sin\varphi_1} = \dfrac{z_2\,m}{2\sin\varphi_2}$

		mean dimensions	outside dimensions
q 31	pitch	$t_m = \left\{\begin{array}{l}\text{as in formula}\\ \text{for } t\text{, q 10}\end{array}\right.$	$t = t_m + \dfrac{b\sin\varphi}{z}\,\pi$
q 32	module	$m_m = \dfrac{t_m}{\pi}$	$m = m_m + \dfrac{b\sin\varphi}{z}$
q 33	width of tooth	$b_m = \beta\,t_m$	$b = \beta\,t$
q 34	pitch circle diameter	$d_m = d - b\sin\varphi$	$d = m\,z$

Axial and radial bearing pressures

q 35 axial bearing pressure $F_a = F\tan\alpha \times \sin\varphi$ N, kgf

q 36 radial bearing pressure $F_r = F\tan\alpha \times \cos\varphi$ N, kgf

q 37 force on tooth profile $F = \dfrac{2\pi M}{t_m\,z}$ N, kgf

---

α : pressure angle (see Q 2)
For other letters see Q 1

MACHINE PARTS

Worm gearing

Worm gearing

Transmission ratio i

q 33 $$i = \frac{n_1}{n_2} = \frac{z_2}{z_1} = \frac{\text{number of teeth per worm gear}}{\text{number of leads of worm}}$$

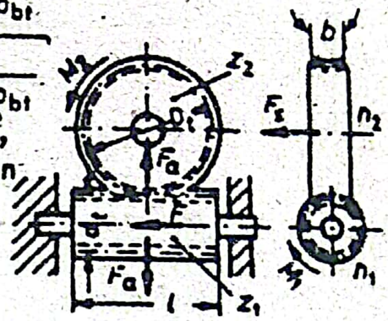
Calculation of worm gear for strength

q 39 pitch $$t\,(\text{mm}) = 97{\cdot}6\sqrt[3]{M_1 \times \frac{1}{\psi \times z_1 \times p_{bt}}}$$

q 40 $$t\,(\text{mm}) = 877\sqrt[3]{\frac{P}{n_1} \times \frac{1}{\psi \times z_1 \times p_{bt}}}$$

M in kgf m, p_{bt} in MN/m^2,
P in H.P., n_1 in revs/min

q 42 length of arc $b = \psi t$



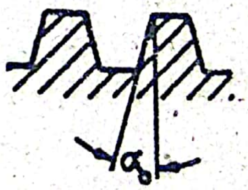
Dimensions

		worm	worm wheel
q 43	module	–	$m = \frac{t}{\pi} = \frac{D_t}{z_2}$
q 44	pitch circle diam.	$d_t = 2\varkappa m$	$D_t = m\,z_2$
q 45	outside diameter	$d_k = d_t + 2\,m$	$D_k = m(z_2 + 2)$
q 46	root circle diam.	$d_f = d_t - 2{\cdot}33\,m$	$D_f = m(z_2 - 2{\cdot}33)$
q 47	length	$l = 2\,m(\sqrt{z_2} + 1)$	
q 48	axial force	$F = 2\,M/D_t$	$F_s = F\tan(\beta + \rho')$
q 49	radial force	$F_a = F\tan\alpha_0$	
q 50	efficiency	$\eta_1 \overset{+)}{=} \frac{\tan\beta}{\tan(\beta + \rho)}$	$\eta_2 \overset{+)}{=} \frac{\tan(\beta - \rho')}{\tan\beta}$

Factor $\varkappa$ at best contact

q 52

number of leads z_1	1	2	3	4
factor $\varkappa$	3·4	4·2	4·8	5·3



q 55 **Irreversible if** $\tan\beta \leq \mu'$

q 56 β : angle of lead $\left(\text{from } \tan\beta = \frac{z_1\,t}{\pi\,d_t} = \frac{z_1}{2\,\varkappa}\right)$
q 57 ρ : angle of friction $(\tan\rho = \mu)$
μ : coefficient of friction (see Z 16)
q 58 μ' : total coeff. of friction $(\mu' = 1{\cdot}034\,\mu = \tan\rho')$
q 59 ψ : 2·5 for cast iron, phosphor bronze and aluminium
+) η_1 where worm, η_2 where worm gear driven
For other letters see Q 1

MACHINE PARTS

Epicyclic gearing

Q 5

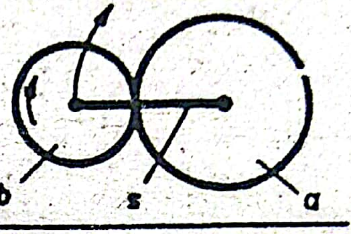
Epicyclic gearing

(transmission ratios)

q 60 $n_a = 0$ (a is fixed)

q 61 $n_s = 1$

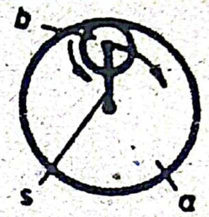
q 62 $n_b = 1 + \frac{a}{b}$



q 63 $n_a = 0$ (a is fixed)

q 64 $n_s = 1$

q 65 $n_b = 1 - \frac{a}{b}$

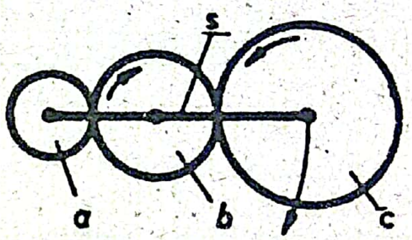


q 66 $n_a = 0$ (a is fixed)

q 67 $n_s = 1$

q 68 $n_b = 1 - \frac{a}{b}$

q 69 $n_c = 1 + \frac{a}{c}$

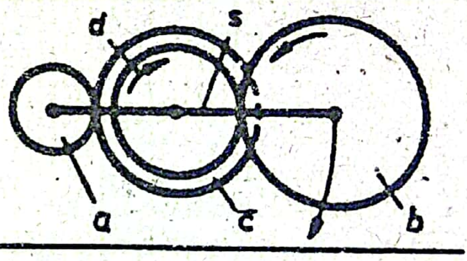


q 70 $n_a = 0$ (a is fixed)

q 71 $n_s = 1$

q 72 $n_c = n_d = 1 + \frac{a}{c}$

q 73 $n_b = 1 - \frac{a\,d}{c\,b}$

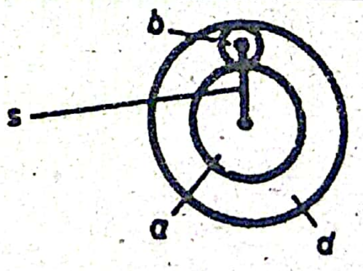


q 74 $n_a = 0$ (a is fixed)

q 75 $n_s = 1$

q 76 $n_d = 1 + \frac{a}{d}$

q 77 $n_b = 1 + \frac{a}{b}$



q 78 $n_d = 0$ (d is fixed)

q 79 $n_s = 1$

q 80 $n_a = 1 + \frac{d}{a}$

q 81 $n_b = 1 - \frac{d}{b}$

$n_d = 0$ (d is fixed)

$n_a = 1$

$n_b = \frac{a}{b}\left(\frac{b-d}{a+d}\right)$

$n_s = \frac{a}{a+d}$

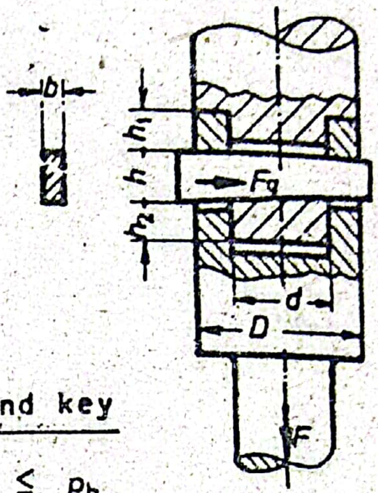
MACHINE PARTS

Key joints

Tapered keys (cotter pins)

Safety margin for load F

In view of the additional load when the key is tightened, a 25 % margin should be added to the following formulae.



Calculating contact pressure p

contact pressure between

rod and key	socket and key
$p = \frac{12.5\,F}{b\,d} \leqq p_b$	$p' = \frac{12.5\,F}{b(D - d)} \leqq p_b$

q 82

Calculating bending stress

q 83 $$h = 0.87\sqrt{\frac{0.625\,F(D + d)}{b\;p_{bt}}}$$

q 84 $$h_1 = h_2 = (0.5 \ldots 0.75)h$$

Calculating shearing stress

q 85 $$\tau = \frac{1.5\,F}{b\,h} \leqq p_q \quad \text{MN/m}^2,\ \text{kgf/mm}^2$$

Force F_q to drive key

for calculation see K 11

Load on screw

core diameter d_k when applied

after screwing operation	during screwing operation
$d_k = \sqrt{\frac{40\,F}{\pi \times p_t}}$ mm	$d_k = \sqrt{\frac{40\,F}{\pi\; 0.75\; p_t}}$ mm

q 86

F : load to be moved contact pressure to be applied in kgf

$p_b - p_q - p_t$: permissible stresses in MN/m^2 (see Z 14)

MACHINE PARTS

Springs

Q 7

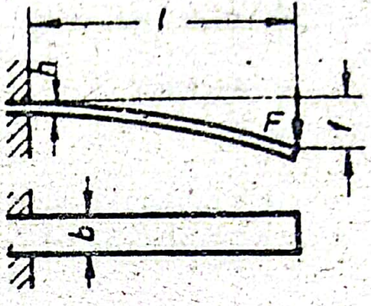
Simple leaf springs

Without preload

q 87 allow. deflect. $f_{max} = \dfrac{2\,l^2\,p_{bt}}{3\,h\,E}$ mm

q 88 allowable load $F_{max} = \dfrac{b\,h^2\,p_{bt}}{6\,l}$ N

q 89 deflection $f = \dfrac{4\,l^3\,F}{b\,h^3\,E}$ mm

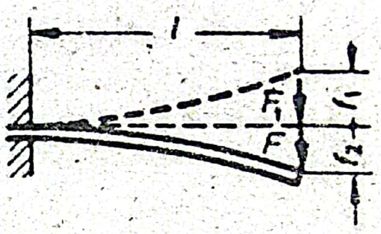


With preload

The above formulae apply to f_{max}, F_{max}, and f as will also total deflection:

q 90 $f = f_1 + f_2 = f_2\,\dfrac{F}{F - F_1}$ mm

q 91 initial deflect. $f_1 = f_2\,\dfrac{F_1}{F - F_1}$ mm

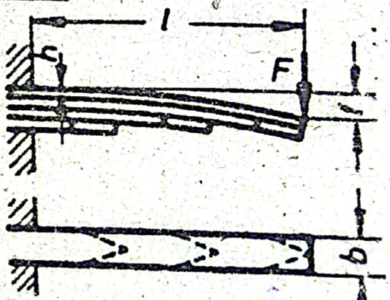


Laminated leaf springs

q 92 allowable deflection $f_{max} = \dfrac{l^2\,\sigma\,p_{bt}}{h\,E}$ mm

q 93 allowable load $F_{max} = \dfrac{N\,b\,h^2\,p_{bt}}{6\,l}$ N

q 94 deflection $f = \dfrac{6\,l^3\,F}{b\,h^3\,N\,E}$ mm



For theoretical shape of individual leaves see p 20 + 2[illegible]

Coiled spring

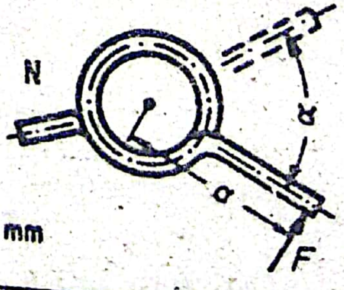
(α is measured in degrees)

q 95 allowable twist $\alpha_{max} = \dfrac{57{\cdot}3\;l\;W_b\;p_{bt}}{I\,E}$

q 96 allowable load $F_{max} = \dfrac{W_b\,p_{bt}}{a}$ N

q 97 twist $\underset{\text{(degrees)}}{\alpha} = \dfrac{57{\cdot}3\;F\,a\,l}{I\,E}$

q 98 length of coil $l = \dfrac{\alpha\,I\,E}{57{\cdot}3\;F\,a}$ mm



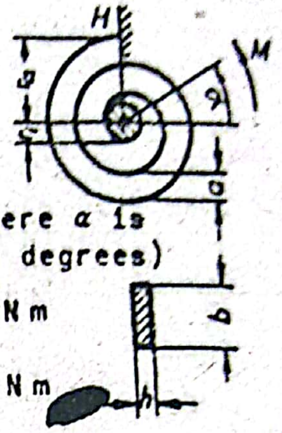
For letters see Q 8

MACHINE PARTS

Springs

Spiral spring

(α is measured in degrees)



q 99 allowable rotation α_{max} (degrees) $= \dfrac{114.6\ l\ p_{bl}}{h\ E}$

q 100 allowable torque $M_{max} = \dfrac{b\ h^2}{6} p_{bl}$ N m

q 101 rotation α (degrees) $= \dfrac{687.6\ M\ l}{E\ b\ h^3}$

q 102 torque $M = \dfrac{\alpha\ E\ b\ h^3}{687.6\ l}$ (where α is in degrees)

q 103 fixed at H $F_1 = \dfrac{M}{r_a}$ N m

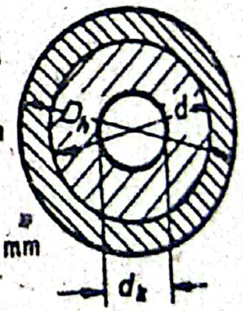
q 104 jointed at H $F_1 = \dfrac{M}{2\ r_a}$ N m

q 105 spacing of coils $a = \dfrac{(r_a^2 - r_i^2)\pi}{l}$ cm, mm

q 106 number of turns $N = \dfrac{l}{\text{mean length of coil}}$

Driving spring

In general, assume diameter of spring casing to be D_h and adjust in calculations.



q 107 core of spring $d_k \approx \dfrac{D_h}{3}$ mm, cm

q 108 thickness of spring $h \approx \dfrac{D_h}{100}$ mm, cm
(round off h to standard thickness)

q 109 length of coil $l = N_1 \pi \dfrac{D_h + d}{2}$ mm

q 110 numb. of turns $n_{max} = N_2 - N_1$

	condition of spring	number of turns	turning moment +)
q 111	unwound	$N_1 = \dfrac{D_h - d}{2\ h}$	$M_{min} \approx \dfrac{E\ b\ h^3\ \pi}{6\ l}\ 3$ N m
q 112	fully wound	$N_2 = \dfrac{d - d_k}{2\ h}$	$M_{max} \approx \dfrac{E\ b\ h^3\ \pi}{6\ l}\ (N_2 - N_1 + 3)$

+) deduct 10 %...30 % for friction

f_2 : working deflect.
N : number of leaves
I : second moment of area

p_{bl} : perm. bending stress } see Z 15
E : modulus of elasticity } see P 3

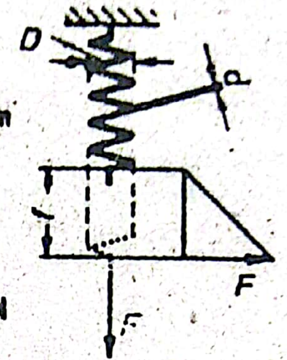
MACHINE PARTS

Springs

Helical spring

(tension and compression springs)

No preload



q 113 diameter ratio $D_m \geqq D - d \geqq 6d$

q 114 diameter of wire $d = \sqrt[3]{\dfrac{F\,8\,D_m}{\pi \cdot p_{qt}}}$ mm

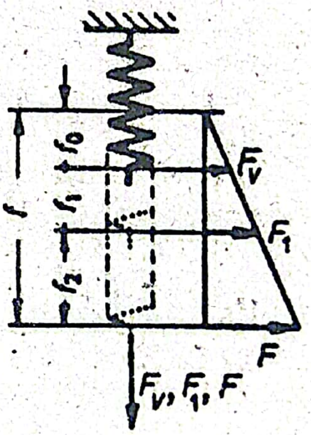
q 115 number of coils $N = \dfrac{f\,d^4\,G}{8\,D_m^3\,F}$

q 116 permiss. load $F_{max} = \dfrac{\pi\,d^3\,p_{qt}}{8\,D}$ N

q 117 deflection $f = \dfrac{8\,N\,D_m^3\,F}{d^4\,G} = \dfrac{\pi\,N\,D^2\,p_{qt}}{d\,G}$ mm

With preload

Use same formulae as above and



q 118 deflection $f = f_0 + f_1 + f_2$

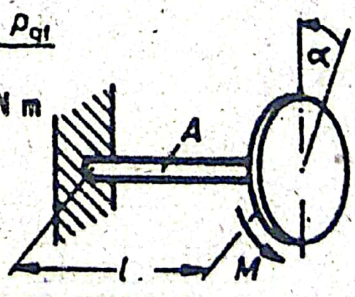
q 119 $= f_2 \dfrac{F}{F - F_1}$

q 120 initial deflection $f_1 = f_2 \dfrac{F_1}{F - F_1}$

q 121 preload $F_v = F \dfrac{f - (f_1 + f_2)}{f}$

Torsion bar

(α is measured in degrees)



q 122 perm. extension $\alpha_{max} = \dfrac{57 \cdot 3\; l\, W_t\, p_{qt}}{I_p\,G}$ (degrees)

q 123 perm torque $M_{max} = W_t \times p_{qt}$ N m

q 124 extension $\alpha = \dfrac{57 \cdot 3\; M\, l}{I_p\,G}$ (degrees)

q 125 torque $M = \dfrac{\alpha\, I_p\, G}{57 \cdot 3\; l}$

f_0 : initial stress developed in winding tension springs
I_p : polar moment of inertia of A see P 7
W_t : torsion stress constant of A see P 7
p_{qt} : permissible torsion stress see Z 15
D_m : mean diameter of helical spring

Shafts

Depending upon the stresses to which they are subjected, shafts should be calculated for:

Permissible stress on material for

q 126 pure torsion $\tau_t = \frac{5{\cdot}1\,M}{d^3} \leqq p_{qt}$ MN/m², kgf/mm²

torsion and bending: see P 7 and P 11.

Distortion due to bending stress

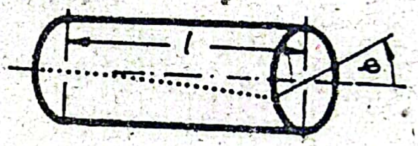
Occurs in electrical machinery through belt, gear or magnetic tension. Where distortion is heavy, the bearings will seize or the airgap will decrease in electrical machines.

Check for bending see P 4.

Distortion due to torsion stress

To ensure even gearchange, particularly with crank shafts, φ should not be excessive.

q 127 $$\underset{\text{(degrees)}}{\alpha} = \frac{180\; l\, M}{\pi\, I_p\, G}$$



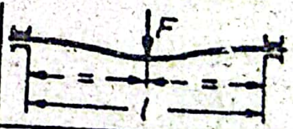
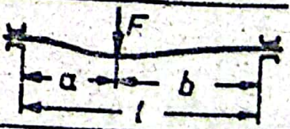
Critical speed n_k determined by torsional vibrations

Most critical for long shafts, widely spaced bearings, and high speeds.

Critical speed n_k may be calculated using M 6.

q.128 Operational speed $n \approx (0{\cdot}80 \ldots 0{\cdot}90)\,n_k$

Critical speed n_b determined by flexural vibrations

q 129	symmetrical load	$n_b\left(\frac{\text{revs}}{\text{sec}}\right) = 1{\cdot}103\sqrt{\frac{g\,E\,I}{F\,l^3}}$	
q 130	asymmetrical load	$n_b\left(\frac{\text{revs}}{\text{sec}}\right) = 0{\cdot}276\sqrt{\frac{g\,E\,I\,l}{F\,a^2\,b^2}}$	

d : shaft diameter | G : modulus of shear see Z 15
E : modulus of elasticity in kgf/mm² see Z 14 + Z 15
I : moment of inertia in mm⁴ see P 3
I_p : polar moment of inertia see P 7
a, b, l in mm; g in mm/s²; F in kgf

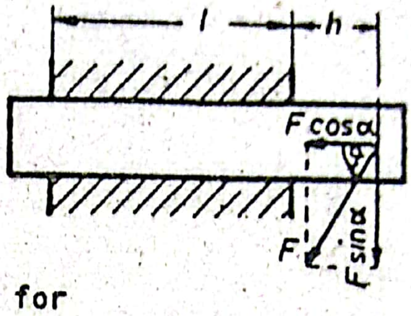
MACHINE PARTS
Cross head guide, Clutchs

Q 11

Crosshead guide

Crosshead guide will operate smoothly only when

q 131 $\tan\alpha < \dfrac{l}{(2h + l)\mu}$; or



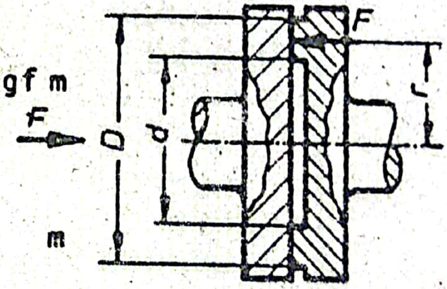
the length ratio is:

q 132 $\dfrac{l}{h} = \lambda > \dfrac{2\mu \tan\alpha}{1-\mu \tan\alpha}$

Where the above conditions for $\tan\alpha$ do not occur, "canting" will occur or the drive will lock.

Friction clutches

Disc clutch

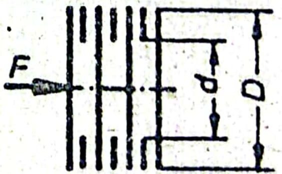


q 133 $M_R = \dfrac{\mu F}{3} \times \dfrac{D^3 - d^3}{D^2 - d^2}$ N m, kgf m

q 134 $\approx \mu F r$

q 135 $r = \dfrac{D + d}{4}$ m

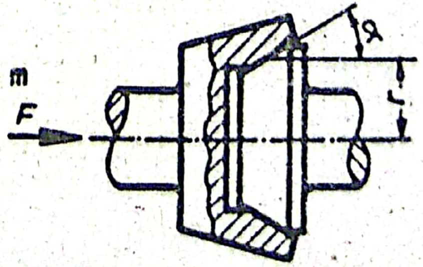
Multi-disc clutch



q 136 $M_R = \dfrac{\mu F N}{3} \times \dfrac{D^3 - d^3}{D^2 - d^2}$ N m, kgf m

q 137 $\approx \mu F r N$

Cone clutch



q 138 $M_R = \dfrac{\mu F r}{\sin\alpha}$ N m, kgf m

To obviate locking; $\tan\alpha$ must $\leq \mu$.

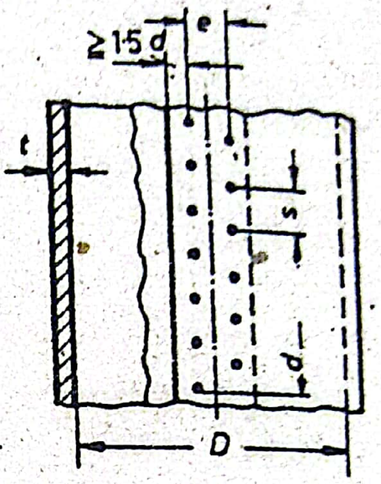
M_R : transmitted torque
r : mean radius

μ : coeff. of friction (see Z16)
N' : number of contact surfaces

MACHINE PARTS

Boilers

Cylindrical boiler



Thickness of wall when

	welded	rivetted
q 139	$t = \frac{p D S}{2\, v\, p_t} + 1\ \text{mm}$	$t = \frac{p D S}{0{\cdot}3\, p_t}$

Dimensions of rivet pattern

q 140 $d = \sqrt{50\, t\ \text{mm}} - 4\ \text{mm}$ | $v = \frac{s - d}{s}$

q 141 $s = 3\, d + 5\ \text{mm}$

q 142 $e = 0{\cdot}8\, s$ | $e_1 = 0{\cdot}6\, s$

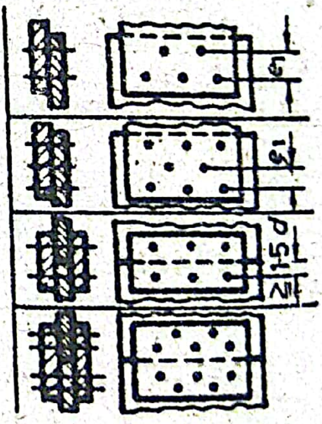
Tensile stresses

	circumferential seam	longitudinal seam	
q 143	$\sigma_t = \frac{\pi D^2 p}{4\, t\, (\pi D - n\, d)}$	$\sigma_t = \frac{D\, p\, s}{2\, (s - d)\, t}$	MN/m², N/mm², kgf/mm²

Plate check $(s - d)\, t\, \frac{p_t}{S} \geq 0{\cdot}5\, D\, p\, t$

Rivet check

q 144	single shear	$2\, \frac{\pi}{4}\, d^2 p_r$	$\geq 0{\cdot}5\, D\, p\, t$
q 145		$3\, \frac{\pi}{4}\, d^2 p_r$	
q 146	double shear	$\frac{\pi}{4}\, d^2 p_r$	
q 147		$3\, \frac{\pi}{4}\, d^2 p_r$	



Permissible shear for rivet joints p_r in N/mm²

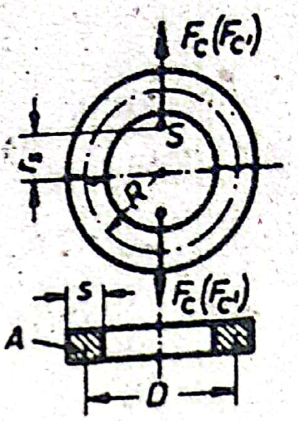
	type of rivetting	single shear	double shear
q 148	single row	60 ... 70	100 ... 120
q 149	two row	55 ... 65	95 ... 115
q 150	three row	50 ... 60	90 ... 110

n : number of rivet holes on a single circumference line
p : internal excess pressure of boiler
p_t : perm. tension stress (Z 15) | S : safety factor (4 ... 5)
D : diameter of boiler | d : diameter of rivet

MACHINE PARTS

Shrunk-on ring

Shrunk-on ring



Where a shrunk-on ring is used to hold together a rotating machine part, e.g. a flywheel etc., the shrinkage force F_c must be at least twice as great as the centrifugal force $F_{c'}$, thus

51 $$F_c \geqq 2 F_{c'}$$

Centrifugal force $F_{c'}$

52 $$F_{c'} = m r_s \omega^2 = 4 \pi^2 m r_s n^2 \qquad \text{N, kgf}$$

53 $$r_s = \frac{2R}{\pi} \qquad \text{m, cm}$$

54 $$m = \frac{G}{g} \qquad \text{kg}$$

Cross section A of shrunk-on ring

55 $$A = \frac{F_c}{2\, p_t} \qquad \text{cm}^2,\ \text{m}^2,\ \text{in}^2$$

Shrinkage allowance λ

λ is the amount by which the internal diameter of the ring must be smaller than the diameter of the machine part to be held together.

56 $$\lambda = \frac{1}{E} D\, p_t$$

Thickness s of a thin shrunk-on ring (boiler formula)

57 $$s = \frac{D\, p_b}{2\, p_t} \qquad \text{m, cm, in}$$

R : mean radius
D : mean diameter [spokes and hub
G : weight of half ring add approximately 5% for
p_b : permissible contact pressure see Z 14
p_t : permissible tensile stress see Z 14

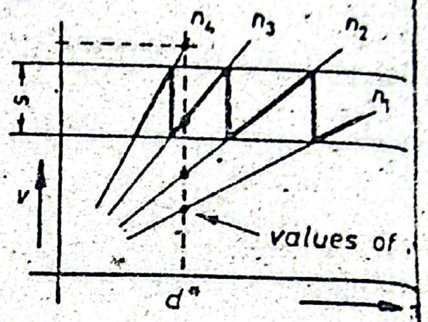
Speed changing

Saw diagram

The saw diagram is used to ascertain speed ch
ging.

Construction of saw diagram

For a random diameter d^* calculate the circumferential velocities v^* corresponding to the given speeds n by $v^* = d^* \pi n$. Plot these values against d^*. Through each of these points and the origin, draw straight lines for each of the speeds n_1, n_2, n_3
Take a convenient value of v and draw a horizo
line L to intersect the lines n_1, n_2, n_3 etc.
each of these points draw a vertical line dow
the next n-line. In the ideal condition, all t
points lie in a horizontal line a distance s b
the line L.



Changing

The best speed ranges are obtained when, as in
diagram shown above, all distances s are equ
A geometric series satisfies this requirement:

r 1 $$n_k = n_1 \varphi^{k-1}$$

The separate speeds are found follows:

r 2 $n_2 = n_1 \varphi$ | $n_3 = n_1 \varphi^2$ | $n_4 = n_1 \varphi^3$ | et

r 3 number of stages $\Big\}$ $$k = 1 + \frac{\lg \frac{n_k}{n_1}}{\lg \varphi}$$

r 4 progressive ratio $\Big\}$ $$\varphi = \sqrt[k-1]{\frac{n_k}{n_1}}$$

standard values of φ: 1·06 – 1·12 – 1
1·41 – 1·58 – 2

n_1 : lowest speed | n_k : highest speed

MACHINE TOOLS

Speed changing

Calculation of speed changes

Cone pulleys

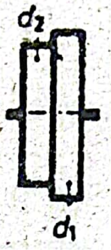
To ensure that the driving belt fits for all speeds in the range, the diameters of the steps on the driving and driven pulleys.

The largest diameter d_1 should be selected. This will be governed by the swing e.g. in lathes.

5

6 $d_1 : d_2 = \sqrt{\varphi} : 1$

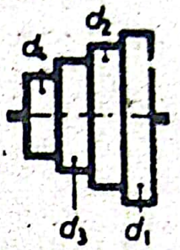
7



$d_1 : d_4 = \sqrt{\varphi^3} : 1$

$d_2 : d_3 = \sqrt{\varphi} : 1$

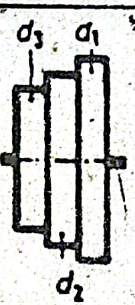
$d_2 = \dfrac{d_1 \sqrt{\varphi}}{2\sqrt{\varphi} - 1}$



8

9 $d_1 : d_3 = \varphi : 1$

10 $d_2 = \dfrac{d_1 + d_3}{2}$

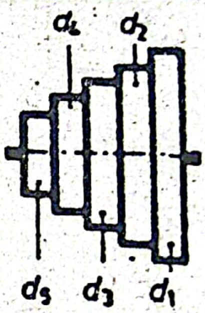


$d_1 : d_5 = \varphi^2 : 1$

$d_2 : d_4 = \varphi : 1$

$d_3 = \dfrac{d_1 + d_5}{2}$

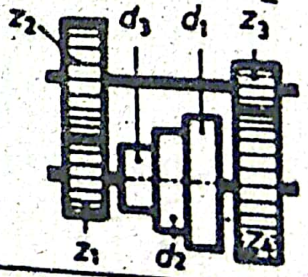
1 $d_4 = \dfrac{d_3 + d_5}{2}$



Toothed wheel counter shaft (transmission ratio i)

12 For tooth wheel calculation use formula $M_{max} = F_b \dfrac{d_1}{2}$

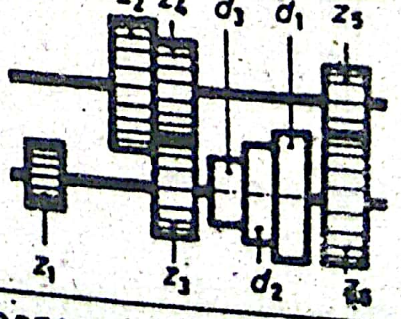
13 $i = \varphi^{k/2} = \dfrac{z_2 \, z_4}{z_1 \, z_3}$



14 $i_1 = \varphi^{k/3} = \dfrac{z_4 \, z_6}{z_3 \, z_5}$

15 $i_2 = \varphi^{2/3 k} = \dfrac{z_2 \, z_6}{z_1 \, z_5}$

16 $z_3 = z_4$ select



M_{max} : max. torque, see R 3

F_b : force of belt

φ : progressive ratio } see R 1

k : number of stages }

MACHINE TOOLS

Calculation of forces

R 3

For machine tools

Drive

r 17 motor output $P_M = \frac{2 \pi r n F_W}{\eta}$

r 18 $= \frac{v F_W}{\eta}$

r 19 shaft diameter $d \approx \sqrt[3]{\frac{5{,}1\, M}{p_{qt}}}$

r 20 torque on working spindle $M = F_W \frac{D}{2}$

Shear resistance F_W for

r 21 lathes $F_W = A k$

r 22 milling machines $F_W = \frac{b t c k}{v}$

r 23 load on milling shaft $F = 1{\cdot}4\, F_W$

Specific resistance k for steel

r 24 $k = a \sigma_B$

tensile strength σ_B in MN/m^2	400	500	600	700	800	900	100
r 25 a	2·5	2·6	2·7	2·8	2·9	3·0	3·2

Driving belt

r 26 force of belt $F_b = \frac{2M}{d} = \frac{P}{v_b}$

r 27 velocity of belt $v_b = d_1 \pi n$

r 28 width of belt from $F_b = B p$

width of belt in mm	B	50	60	70	80	100	120	15
r 29 perm. load in N/mm^2	p	10	12	12	13	14	15	1

For symbols see R 4.

MACHINE TOOLS

Power required

R 4

Estimated power required

	type of machine			power required in kW	
r30	centre lathes	light		0·015 x swing	in mm
r31		medium		0·030 x swing	in mm
r32		heavy		0·045 x swing	in mm
r33	chuck lathes	light		0·075 x stock feed	in mm
r34		heavy		0·120 x stock feed	in mm
r35	automatic lathes	single	spindle	as for chuck lathes	
r36		multi-		as for chuck lathes x number of operating spindles	
r37	drilling machines	standard		0·075 x drill diameter	in mm
r38		high perf.		0·150 x drill diameter	
r39	milling machines	light		0·075 x table area	in mm^2
r40		heavy		0·150 x table area	in mm^2
r41	shaping			0·008 x greatest stroke	in mm
r42	long plane machines			0·006 x width of plane	
r43	cylindrical grinding			0·020 x grinding wheel diameter	in mm
r44	surface grinding			0·010 x grinding wheel diameter	in mm

Symbols for R 3

A : chip section
b : width of milling edge
c : feed
d_1 : pulley diameter
D : diameter of material for lathes
diameter of cutter for milling machines
p : permissible belt load
t : depth of cut
v : cutting velocity
η : efficiency of drive from motor to operating spindle
p_{qt}: permissible torsional shear stress (see Z 14)